Quiz #2

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can’t make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will lose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

Problem: Consider the linear system $Ax = b$ where

\[
A = \begin{pmatrix}
1 & -1 & -2 & -2 & -2 \\
3 & -2 & -2 & -2 & -2 \\
-3 & 2 & 1 & 1 & -1
\end{pmatrix}
\]

and

\[
b = \begin{pmatrix}
3 \\
-1 \\
-1
\end{pmatrix}
\]

1. Solve the system and write the general solution in parametric-vector form.

2. Deduce a geometric description of the set of solutions.

3. Give a particular solution of the system.

4. Write the solution set for the homogeneous equation $Ax = 0$, parametrically and geometrically as before. (Use previous questions)

5. Does the columns of $A$ span $\mathbb{R}^3$? (Use previous questions)

6. Is $c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in the span of the column of $A$? (Use previous questions)

7. Let $T : \mathbb{R}^5 \to \mathbb{R}^3$ be the matrix transformation $x \mapsto Ax$.

(a) Compute the image of $a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ by $T$?
(b) Is \( d = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \) in the range of \( T \)? (Use previous questions)

(c) Describe the range of \( T \) as a spanning set.

Solution:

1. We need to reduce the augmented matrix. I’ll leave the details to you. 

\[
\begin{pmatrix}
1 & -1 & -2 & -2 & 3 \\
3 & -2 & -2 & -2 & -1 \\
-3 & 2 & 1 & 1 & -1 & -1
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 0 & 0 & 0 & -1 & 4 \\
0 & 1 & 0 & 0 & -8 & -18 \\
0 & 0 & 1 & 1 & 3 & 2
\end{pmatrix}
\]

We see that \( x_1, x_2 \), and \( x_3 \) are basic variables and \( x_4 \) and \( x_5 \) are free variables. We rewrite the system, i.e., we get

\[
\begin{align*}
x_1 &= -11 + 4x_4 \\
x_2 &= -18 + 8x_4 \\
x_3 &= 2 - x_4 - 3x_5
\end{align*}
\]

Then we find that the parametric vector form of the general solution set is

\[
S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -11 \\ -18 \\ 2 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 4 \\ 8 \\ -3 \\ 0 \\ 1 \end{pmatrix} : x_4, x_5 \in \mathbb{R} \right\}
\]

2. Geometrically, the solution set is a plane passing through the point \( \begin{pmatrix} -11 \\ -18 \\ 2 \\ 0 \\ 0 \end{pmatrix} \) and with direction \( \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 4 \\ 8 \\ -3 \\ 0 \\ 1 \end{pmatrix} \).

3. A particular solution of the system is

\[
\begin{pmatrix} -11 \\ -18 \\ 2 \\ 0 \\ 0 \end{pmatrix} \quad (x_4 = x_5 = 0)
\]

4. One can deduce from what have been done in class that the parametric vector form of the general solution set of \( Ax = 0 \) is

\[
S_0 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = +x_4 \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 4 \\ 8 \\ -3 \\ 0 \\ 1 \end{pmatrix} : x_4, x_5 \in \mathbb{R} \right\}
\]
and geometrically, the solution set is a plane passing through the origin and with
direction \[
\begin{pmatrix}
0 \\
0 \\
-1 \\
1 \\
0
\end{pmatrix}
\] and
\[
\begin{pmatrix}
4 \\
8 \\
-3 \\
0 \\
1
\end{pmatrix}.
\]

5. We have observed that the row reduced form of \( A \) is
\[
\begin{pmatrix}
1 & 0 & 0 & 0 & -4 \\
0 & 1 & 0 & 0 & -8 \\
0 & 0 & 1 & 1 & 3
\end{pmatrix}
\] from the first question so we note that there is a pivot position in each row thus from the spanning theorem the column of \( A \) span \( \mathbb{R}^3 \).

6. By definition, since the column of \( A \) do span \( \mathbb{R}^3 \) thus \( c = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) is in the span of the column of \( A \).

7. (a) The image of \( a = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \) by \( T \) is
\[
T(a) = Aa = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}
\]

(b) \( d = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} \) is in the range of \( T \) if and only if he equation \( T(x) = d \) that is \( Ax = d \) has a solution that is equivalent to say that \( d \) is in the span of the columns of \( A \) which is the case as explained before.

(c) The range of \( T \) equals the span of the columns of \( A \).