Problem Set #1

Due Friday 12 September in Recitation

Exercise 1: (∗)
Let \( a \in \mathbb{Z} \). Prove that \( \gcd(2a + 3, a + 2) = 1 \).

Exercise 2:
Let \( a, b \in \mathbb{Z} \) and \( \gcd(a, b) = 1 \). Prove that \( \gcd(a + 3b, a + b) = 1 \) or 2.

Exercise 3: (∗)
Prove that if \( a \) and \( b \) are positive integers with \( \gcd(a, b) = 1 \), then \( \gcd(a^2, b^2) = 1 \).

Exercise 4: (∗)
Show that \( 5 \mid n^5 - n \), for all integers \( n \). (Hint: Five Different Cases)

Exercise 5:(*)
Prove using induction on \( n \) that 4 divides \( 1 + 3^{2n+1} \) for all \( n \in \mathbb{N} \).

Exercise 6: (∗ ∗)
Prove that for any positive integer \( x \), there exist a non-negative integer \( k \) and \( b_i \in \{0, 1\} \), where \( 0 \leq i \leq k \), such that
\[
x = b_k 2^k + b_{k-1} 2^{k-1} + \cdots + b_1 2 + b_0.
\]
Prove that this expression is unique. (Hint: Use induction on \( x! \))