FINAL EXAM (1h50)
Maximal Score: 200 points

Show ALL steps and make sure I understand how you get the answer to have full credit! No material allowed!

Wednesday december 18th

Problem 1: (⋆) 10 points
Let \(a = 2^4 \cdot 13 \cdot 2^1 \cdot 19\) and \(b = 2^3 \cdot 5^2 \cdot 13\). Give the prime factorization of \(\gcd(a^2, b^3)\)

Problem 2: (⋆) 10 points
Show that \(4(29!) + 5!\) is divisible by 31. (Hint: Wilson’s theorem!)

Problem 3: (⋆) 10 points
Give a non-trivial factor of \(2^{52} + 1\).

Problem 4: (⋆) 10 points
Find all right angled triangles with coprime integer sides and base of length 28. (Hint: Pythagorean triples!)

Problem 5: (⋆) 10 points
Prove that \(\sqrt{7}\) is irrational. (Hint: By contradiction!)

Problem 6: (⋆) 10 points
Suppose that \(n^2 = \sum_{d|n} f(d)\). Evaluate \(f(8)\).

Problem 7: (⋆) 15 points
You have chosen to do RSA cryptography with modulus \(n = pq\) where \(p = 7\) and \(q = 19\)

1. Compute the least common multiple \([\phi(p), \phi(q)]\).
2. Suppose that the encode exponent is \(e = 5\). Calculate a decode exponent \(d\).

Problem 8: (⋆) 20 points

1. Use the Euclidean algorithm to compute the greatest common divisor \((263, 271)\).
2. Solve the linear equation \(263x - 271y = 5\) or explain why there are no solutions.

Problem 9: (⋆) 10 points
Solve the simultaneous congruences equations :

\[
\begin{align*}
    x &\equiv 5 \mod 7 \\
    x &\equiv 2 \mod 5
\end{align*}
\]
Problem 10: (⋆) 20 points Say if the following Gaussian integers are prime and when they are not give a prime factorization in \( \mathbb{Z}[i] \):

1. 7
2. 1 + 2i

Problem 11: (⋆) 15 points

1. Evaluate \( \phi(1500) \).
2. Compute the remainder when \( 7^{1203} \) is divided by 1500.

Problem 12: (⋆) 10 points Use induction to prime that \( 6^n \equiv 5n + 1 \mod 25 \) for all positive integer \( n \).

Problem 13: (⋆) 20 points

1. Find the continued fraction expansion of \( \sqrt{30} \).
2. Find the quadratic \( \alpha \) with continued fraction expansion \( \alpha = [2, 3] \).

Problem 14: (⋆) 30 points

1. Evaluate \( \left( \frac{293}{331} \right) \).
2. Prove that the Diophantine equation \( x^2 + y^2 = 12z + 7 \) does not have any solution.
   (Hint: modulo 12.)
3. Prove that the quadratic congruence \( x^2 - 4xy + 5y^2 \equiv 0 \mod 11 \) has no solution.

\(^1(⋆) = \text{easy}, (⋆★) = \text{medium}, (⋆★★) = \text{challenge}\)