Homework #2

Exercises

1. Mark each statement True or False. Justify each answer.
(a) The symbol $\forall$ means "for every."
True, this is one of the replacements for $\forall$.
(b) The negation of a universal statement is another universal statement.
False the negation of $\forall$ is $\exists$ and the negation of $\exists$ is $\forall$.

2. Mark each statement True or False. Justify each answer.
(a) The symbol "$\exists$" means "there exist several"
False, "$\exists$" means "there exist at least one"
(b) If a variable is used in the antecedent of an implication without being quantified, then the universal quantifier is assumed to apply.
True, if you are working with a proposition that has variables and the antecedent has no quantifier then you apply the universal quantifier.
(c) The order in which quantifiers are used affects the truth value.
True, since the $\forall$ and $\exists$ quantifiers have different definitions the order in which they are used affects the truth value.
Example:
$\forall x \in \mathbb{R} \exists y \in \mathbb{R}$ such that $\forall \epsilon > 0, |x - y| < \epsilon$, True
$\exists y \in \mathbb{R}$ such that $\forall x \in \mathbb{R}, \forall \epsilon > 0 |x - y| < \epsilon$, False

3. Write the negation of each statement.
(a) Some pencils are red.
Statement in mathematics: At least one pencil is red.
All pencils are not red.
(b) All chairs have four legs.
There exist a chair that does not have four legs.
(c) No one on the basketball team is over 6 feet 4 inches tall.
Statement in mathematics: For all members of the basketball team no one is 6 feet 4 inches tall.
There exist a member of the basketball team such that he/she is 6 feet 4 inches tall.
tall.
(d) \( \exists x > 2 \) such that \( f(x) = 7 \).
\( \forall x > 2, f(x) \neq 7 \).
(e) \( \forall x \in A, \exists y > 2 \) such that \( 0 < f(y) < f(x) \).
\( \exists x \in A \) such that \( \forall y > 2 \ f(y) \geq 0 \) or \( f(y) \geq f(x) \).
(f) If \( x > 3 \), then \( \exists \varepsilon > 0 \) such that \( x^2 > 9 + \varepsilon \).
\( x > 3, \) and \( \forall \varepsilon > 0, x^2 \leq 9 + \varepsilon \).

4. Write the negation of each statement.
(a) Everyone likes Robert.
\textit{Statement in mathematics: All members like Robert.}
\textbf{At least one student does not like Robert.}
(b) Some students work part time.
\textit{Statement in mathematics: At least one student works part time}
\textbf{All Students do not work part time.}
(c) No square matrices are triangular.
\textit{Statement in mathematics: All square matrices are not triangular.}
\textbf{There exist a square matrix that is triangular.}
(d) \( \exists x \in B \) such that \( f(x) > k \).
\( \forall x \in B, f(x) \leq k \).
(e) If \( x > 5 \), then \( f(x) < 3 \) or \( f(x) > 7 \).
\( x > 5 \) and \( f(x) \geq 3 \) and \( f(x) \leq 7 \).
(f) \( x \) is in \( A \), then \( \exists y \) in \( B \) such that \( f(x) < f(y) \).
if \( x \in A, \) and \( \forall y \in B, f(x) \geq f(y) \).

5. Determine the truth value of each statement, assuming \( x \) is a real number. Justify your answer.
(a) \( \exists x \) in the interval \([2, 4]\) such that \( x < 7 \).
\textit{What is this statement saying in English? There is at least one \( x \) in the interval \([2,4]\) and \( x < 7 \).}
\textbf{True, Take} \( x:=3, x:=3 \) satisfies \( x \in [2, 4] \) and \( x < 7 \).
(b) \( \forall x \) in the interval \([2, 4]\), \( x < 7 \).
\textit{What is this saying in English? For all \( x \)'s in the interval \([2,4]\) \( x < 7 \)}
\textbf{True, Let} \( x \) in the interval \([2, 4]\) \( 2 \leq x \leq 4 < 7, x < 7 \)
(c) \( \exists x \in R \) such that \( x^2 = 5 \).
\textit{What is this statement saying in English? There exist a least one \( x \) such that if you solve for \( x \) in \( x^2 = 5, x \in R \).}
\textbf{True, Take} \( x:= \sqrt{5}, x:= \sqrt{5} \) satisfies \( x \in R \) and \( x^2 = 5 \)
(d) \( \forall x \in R, x^2 = 5 \).
\textit{What is this statement saying in English? For all \( x \), you can solve \( x^2 = 5, x \in R \).}
\textbf{False, take} \( x:=2, x^2=4, x:=2 \) satisfies \( x \in R \) and \( x^2 \neq 5 \).
(e) \( \exists x \in \mathbb{R} \) such that \( x^2 \neq -3 \).

What is this statement saying in English? There exist at least one value in \( x \in \mathbb{R} \) such that \( x^2 \neq -3 \)

True, take \( x := 2 \), \( x^2 = 4 \), \( x := 2 \) satisfies \( x \in \mathbb{R} \) and \( x^2 \neq -3 \)

(f) \( \forall x, x^2 \neq -3. \)

What is this statement saying in English? For all values of \( x \in \mathbb{R}, x^2 \neq -3 \)

True, Let \( x \in \mathbb{R}, x^2 \geq 0 \) so since \( x^2 = -3 < 0 \) we can not have any solution in \( \mathbb{R} \)

(g) \( \exists x \in \mathbb{R} \exists x \div x = 1 \)

What is this statement saying in English? There is at least one \( x \in \mathbb{R} \) such that if you divide it by itself it will give you one

True, Take \( x := 5 \), \( x := 5 \) satisfies \( x \in \mathbb{R} \) and \( x \div x = 1 \).

(h) \( \forall x, x \times x = 1 \)

For all x’s \( x \in \mathbb{R} \) you can divide the number by itself and get one

False, Take \( x := 0 \), \( x := 0 \) satisfies that \( x \in \mathbb{R} \) and \( x \div x \neq 1 \)

To determine the truth value of the following statements using the definition of quantifiers and negation. If its true for \( \exists \) you find one particular example that makes it true. If its false for \( \exists \), you do the negation so that means you prove it for \( \forall \). If its true for \( \forall \), you just prove the statement using Let. If it is false for \( \forall \), you prove by doing the negation \( \exists \) and finding a particular example which makes it false. This is seen throughout these examples.

6. Determine the truth value of each statement, assuming \( x \) is a real number. Justify your answer.

(a) \( \exists x \) in the interval \( [3,5] \) such that \( x \geq 4 \).

English Translation: There is at least one \( x \in \mathbb{R} \), \( x \in [3,5] \) such that \( x \geq 4 \).

True, Take \( x := 4 \), \( x := 4 \) satisfies \( x \in [3,5] \) and \( x \geq 4 \)

(b) \( \forall x \) in the interval \( [3,5] \), \( x \geq 4 \).

For all \( x \)‘s in the interval \( [3,5] \), \( x \geq 4 \)

False, Take \( x := 3 \), \( x := 3 \) satisfies \( x \in [3,5] \) and \( x < 4 \)

(c) \( \exists x \) \( x \in \mathbb{R} \) such that \( x^2 \neq 3 \).

English Translation: There is at least one \( x \in \mathbb{R} \) that makes \( x^2 \neq 3 \) is true.

True, take \( x := 2 \), \( x^2 = 4 \), \( x := 2 \) satisfies \( x \in \mathbb{R} \) and \( x^2 \neq 3 \)

(d) \( \forall x \in \mathbb{R} \), \( x^2 \neq 3 \).

English translation: For all x’s in the \( \mathbb{R} \) \( x^2 \neq 3 \) is true.

False, Take \( x := \sqrt{3}, x := \sqrt{3} \) satisfies \( x \in \mathbb{R} \), and \( x^2 = 3 \).

(e) \( \exists x \in \mathbb{R} \) such that \( x^2 = -5 \).

English translation: There is at least on \( x \) in the \( \mathbb{R} \) that makes this equation true.

False, Let \( x \in \mathbb{R}, x^2 > 0, x^2 = -5 < 0 \). Therefore we cannot have a solution in \( \mathbb{R} \)

(f) \( \forall x \in \mathbb{R}, x^2 = -5 \).

For all \( x \in \mathbb{R} \) this equation is true

False, Take \( x := 2 \), \( x^2 = 4 \) \( x := 2 \) satisfies \( x \in \mathbb{R} \), and \( x^2 \neq -5 \)

(g) \( \exists x \in \mathbb{R} \) such that \( x-x=0 \).
There is at least one element in \( \mathbb{R} \) that makes \( x - x = 0 \) true

True. Take \( x := 5 \), \( x := 5 \) satisfies \( x \in \mathbb{R} \) and \( x - x = 0 \)

(h) \( \forall x \in \mathbb{R}, x - x = 0 \).

English translation: For all \( x \)'s in the \( \mathbb{R} \), this equation is true.

True. Let \( x \in \mathbb{R} \), whenever \( x > 0 \), \( x < 0 \), or \( x = 0 \), \( x - x = 0 \)

7. Below are two strategies for determining the truth value of a statement involving a positive number \( x \) and another statement \( P(x) \).

For each statement below, indicate which strategy is appropriate.

(i) Find some \( x > 0 \) such that \( P(x) \) is true.
(ii) Let \( x \) be the name for any number greater than 0 and show that \( P(x) \) is true.
(a) \( \forall x > 0, P(x) \). (ii)
(b) \( \exists x > 0 \) such that \( P(x) \). (i)
(c) \( \exists x > 0 \) such that \( \neg P(x) \). (ii)
(d) \( \forall x > 0, \neg P(x) \). (i)

8. Which of the following best identifies \( f \) as a constant function where \( x \) and \( y \) are real numbers.

**Constant Function:** The output value is the same no matter what the input value is.

(a) \( \exists x \in \mathbb{R} \) such that \( \forall y \in \mathbb{R}, f(x) = y \) No

This is telling us that the image of \( x \) of \( \mathbb{R} \), which is not compatible with the definition of a map from \( \mathbb{R} \).

(b) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \) such that \( f(x) = y \) No

This is just telling you that \( f \) is a map from \( \mathbb{R} \) to \( \mathbb{R} \).

(c) \( \exists y \in \mathbb{R} \) such that \( \forall x \in \mathbb{R}, f(x) = y \) Yes

This is the definition of subjectivity.

(d) \( \forall y \in \mathbb{R}, \exists x \in \mathbb{R} \) such that \( f(x) = y \) No

This is the definition of subjectivity.

9. Determine the truth value of each statement, assuming that \( x \) and \( y \) are real numbers. Justify your answer.

(a) \( \forall x, y \in \mathbb{R}, x \leq y \).

False, since the original statement is false we will prove the negation of the original statement.

The negation is: \( \exists x, y \in \mathbb{R}, x > y \).

Take: \( x := 5 \)

Take \( y := 4 \)

\( x \) and \( y \) both satisfy that \( x \in \mathbb{R} \) and that \( x > y \). This is contradictory to the original statement.

(b) \( \exists x \) and \( y \in \mathbb{R} \) such that \( x \leq y \).

True
Take \( x := 3 \)
Take \( y := 4 \)
x and \( y \) both satisfy that \( x, y \in \mathbb{R} \) and that \( x \leq y \)

(c) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \) such that \( x \leq y \).
True

Draft:
Let \( x \in \mathbb{R} \)
We can find a \( y \in \mathbb{R} \) such that \( x \leq y \)
Proof:
Let \( x \in \mathbb{R} \)
Take \( y := x + 1 \)
Therefore \( x \leq y \). Notice how the above problem could’ve been solved by taking \( y := x \) because both values can be equal. It is easier to equate the variables when possible instead of finding another example.
The difference between this problem and the problems we have done before is that this one involves two different quantifiers. As a result, it is easy to get confused on what you are evaluating. It is essential to do a draft before the proof.

(d) \( \exists x \in \mathbb{R} \) such that \( \forall y \in \mathbb{R} \) \( x \leq y \).
False, the original statement is false because \( x \) is independent and this is not true for all \( y \)'s if \( x \) is set beforehand.
Since the original statement is false we will prove the negation of the original statement.
The negation is: \( \forall x \in \mathbb{R} \exists y \in \mathbb{R} \) such that \( x > y \)
Proof:
Let \( x \in \mathbb{R} \)
Take \( y := x - 1 \)
Then \( x > y \)
Which is contradictory to the original statement we started with.

10. Determine the truth value of each statement, assuming that \( x \) and \( y \) are real numbers. Justify your answer.
(a) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \) such that \( xy = 0 \)
True

Draft:
For all \( x \in \mathbb{R} \)
We can find a \( y \in \mathbb{R} \) such that \( xy = 0 \)
Proof:
Let \( x \in \mathbb{R} \)
Take \( y := 0 \)
then clearly \( xy = 0 \)
(b) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \) such that \( xy = 1 \)

The original statement is false because when 0 is multiplied with another \( \mathbb{R} \) the product can never be equal to one.

Since the original statement is false we will prove the negation of the original statement.

The negation is: \( \exists x \in \mathbb{R} \) such that \( \forall y \in \mathbb{R} \) \( xy \neq 1 \)

Draft:

\[
\text{Given an } x
\]

For all \( y \)'s this statement \( xy \neq 0 \) is true

Proof:

\[
\text{Take } x := 0
\]

Let \( y \in \mathbb{R} \)

then clearly \( xy \neq 1 \)

(c) \( \exists y \in \mathbb{R} \) such that \( \forall x \in \mathbb{R}, xy = 1 \)

False, the original statement is False. Since the original statement is false we will prove the negation of the original statement.

The negation is: \( \forall y \in \mathbb{R} \exists x \in \mathbb{R} \) such that \( xy \neq 1 \)

Proof:

\[
\text{Let } y \in \mathbb{R}
\]

\[
\text{Take } x := 0
\]

Then clearly \( xy \neq 1 \)

(d) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \) such that \( xy = x \)

False, the original statement is false. Since the original statement is false we will prove the negation of the original statement.

The negation is: \( \exists x \in \mathbb{R} \forall y \in \mathbb{R} \) such that \( xy \neq x \)

Proof:

\[
\text{Take } x := 1
\]

Let \( y \in \mathbb{R} \)

then clearly \( xy \neq x \)

11. Determine the truth value of each statement, assuming that \( x, y, \) and \( z \) are real numbers. Justify your answer.

(a) \( \exists x \in \mathbb{R} \) such that \( \forall y \in \mathbb{R} \exists z \in \mathbb{R} \) such that \( x + y = z \)

True,

\[
\text{Proof:
}\]

\[
\text{Take } x := 0
\]

Let \( y \in \mathbb{R} \)

Take \( z := y \)

Then clearly \( x + y = z \)

(b) \( \exists x \) and \( y \in \mathbb{R} \) such that \( \forall z \in \mathbb{R}, x + y = z \)

False, the original statement is false. Since the original statement is false we
will prove the negation of the original statement.
The Negation is: \( \forall x \text{ and } y \in \mathbb{R} \exists z \in \mathbb{R}, x+y \neq z \)
Proof:
True
Proof: Let \( x, y \in \mathbb{R} \)
Take \( x := x+y+1 \)
then clearly \( x+y \neq z \)
(c) \( \forall x \text{ and } y \in \mathbb{R}, \exists z \in \mathbb{R} \text{ such that } y - z = x \)
Let \( x, y \in \mathbb{R} \)
Take \( z := -x + y \)
Then clearly \( y - z = x \)
(d) \( \forall x \text{ and } y \in \mathbb{R}, \exists z \in \mathbb{R} \text{ such that } xz = y \)
False, The original statement is false. Since the original statement is false so we will prove the negation of the original statement.
The negation is: \( \exists x \text{ and } y \in \mathbb{R} \text{ such that } \forall z \in \mathbb{R} xz \neq y \)
Proof
Take \( x := 0 \)
Take \( y := 2 \)
Let \( z \in \mathbb{R} \)
Then clearly \( xz \neq y \)
(e) \( \exists x \in \mathbb{R} \text{ such that } \forall y, z \in \mathbb{R}, z > y \text{ implies that } z > x+y \)
True,
Draft:
We want to find an \( x \in \mathbb{R} \) such that
For all \( y, z \in \mathbb{R} \)
Suppose \( z > y \)
We want to prove that \( z > x+y \)
Proof:
Take \( x := 0 \)
Let \( x, y \in \mathbb{R} \)
Suppose \( z > y \)
Then clearly \( z > x+y \).
(f)\( \forall x \in \mathbb{R}, \exists y, z \in \mathbb{R} \text{ such that } z > y \text{ implies that } z > x+y \)
True,
Draft:
For all \( x \in \mathbb{R} \)
We want to find a \( y, z \in \mathbb{R} \)
Suppose \( z > y \)
We want to prove that \( z > x+y \)
Proof:
Let \( x \in \mathbb{R} \)
Take \( y := 0 \)
Take \( z := x + 1 \)
Suppose \( z > y \)
Then clearly \( z > x + y \).

12. 11. Determine the truth value of each statement, assuming that \( x, y, \) and \( z \) are real numbers. Justify your answer.

(a) \( \forall x \text{ and } y \in \mathbb{R}, \exists z \in \mathbb{R} \text{ such that } x + y = z \)
True,
Let \( x, y \in \mathbb{R} \)
Take \( z := x + y \)
Then clearly \( x + y = z \)

(b) \( \forall x \in \mathbb{R} \exists y \in \mathbb{R} \text{ such that } \forall z \in \mathbb{R}, x + y = z \)
False, The original statement is false. Since the original statement is false we will prove by negating the original statement.
The negation is: \( \exists x \in \mathbb{R} \text{ such that } \forall y \in \mathbb{R} \exists z \in \mathbb{R} \text{ such that } x + y \neq z \)
Take \( x := 0 \)
Let \( y \in \mathbb{R} \)
Take \( z := y + 1 \)
then clearly \( x + y \neq z \)

(c) \( \exists x \in \mathbb{R} \text{ such that } \forall y \in \mathbb{R}, \exists z \in \mathbb{R} \text{ such that } xz = y \)
True,
Take \( x := 1 \)
Let \( y \in \mathbb{R} \)
Take \( z := y \)
Then clearly \( xz = y \)

(d) \( \forall x \text{ and } y \in \mathbb{R}, \exists z \in \mathbb{R} \text{ such that } yz = x \)
False, the original statement is false. Since the original statement is false we will prove the negation of the original statement.
The negation is: \( \exists x \text{ and } y \in \mathbb{R} \text{ such that } \forall z \in \mathbb{R}, yz \neq x \)
Proof: Take \( x := 1 \)
Take \( y := 0 \)
Let \( z \in \mathbb{R} \)
Then clearly \( yz \neq x \)

(e) \( \forall x \in \mathbb{R}, \exists y \in \mathbb{R} \text{ such that } \forall z \in \mathbb{R}, z > y \text{ implies that } z > x + y \)
False, the original statement is false. Since the original statement is false we will prove the negation of the original statement.
The negation is: \( \exists x \in \mathbb{R} \text{ such that } \forall y \in \mathbb{R} \exists z \in \mathbb{R}, z > y \text{ and } z \leq x + y \).
Proof: Take \( x := 0 \)
Let \( y \in \mathbb{R} \)
Take \( z := y + 1 \)
Then \( z > y \) and \( z \leq x+y \). So both statements not true at the same time.

(f) \( \forall x, y \in \mathbb{R}, \exists z \in \mathbb{R} \) such that \( z > y \) implies that \( z > x+y \).

True

Draft:
For all \( x, y \in \mathbb{R} \)
We can find a \( z \in \mathbb{R} \)
Suppose that \( z > y \)
We want to prove that \( z > x+y \)
Proof:
Let \( x, y \in \mathbb{R} \)
Take \( z := x+y+1 \)
Suppose \( z > y \)

For the next few questions
(a) Rewrite the defining conditions in logical symbolism using \( \forall, \exists \Rightarrow \)
(b) Write the negation

13. A function \( f \) is even if for every \( x \), \( f(-x) = f(x) \)
(a) \( \forall x \in \mathbb{R}, f(-x) = f(x) \)
(b) \( \exists x \in \mathbb{R} \) such that \( f(-x) \neq f(x) \)

14. A function \( f \) is periodic if there exists a \( k > 0 \) such that for every \( x \), \( f(x+k) = f(x) \)
(a) \( \exists k > 0 \) such that \( \forall x \in \mathbb{R}, f(x+k) = f(x) \)
(b) \( \forall k > 0, \exists x \in \mathbb{R} \) such that \( f(x+k) \neq f(x) \)

15. A function \( f \) is increasing if for every \( x \) and \( y \), if \( x \leq y \), then \( f(x) \leq f(y) \)
(a) \( \forall x, y \in \mathbb{R}, x \leq y \Rightarrow f(x) \leq f(y) \)
(b) \( \exists x, y \in \mathbb{R}, x \leq y \) and \( f(x) > f(y) \)

16. A function \( f \) is strictly decreasing if for every \( x \) and \( y \), if \( x < y \), then \( f(x) > f(y) \)
(a) \( \forall x, y \in \mathbb{R}, x < y \Rightarrow f(x) > f(y) \)
(b) \( \exists x, y \in \mathbb{R}, x < y \) and \( f(x) \leq f(y) \)

17. A function \( f : A \rightarrow B \) is injective if for every \( x \) and \( y \) in \( A \), \( f(x) = f(y) \), then \( x = y \)
(a) \( \forall x, y \in A, f(x) = f(y) \Rightarrow x = y \)
(b) \( \exists x, y \in A, f(x) = f(y) \) and \( x \neq y \)

18. A function \( f : A \rightarrow B \) is surjective if for every \( y \) in \( B \) there exists an \( x \) in \( A \), such that \( f(x) = y \)
(a) \( \forall y \in B, \exists x \in A \) such that \( f(x) = y \)
(b) \( \exists y \in B, \) such that \( \forall x \in A, f(x) \neq y \)

19. A function \( f : D \rightarrow \mathbb{R} \) is continuous at \( c \in D \) if for every \( E > 0 \) there is a \( \delta > 0 \) such that \( |f(x) - f(c)| < E \) whenever \( |x - c| < \delta \) and \( x \in D \).
(a) \( \forall E > 0, \exists \delta > 0 \) such that \( |f(x) - f(c)| < E \Rightarrow |x - c| < \delta \) and \( x \in D \).
(b) \( \exists E > 0, \) such that \( \forall \delta > 0 \) such that \( |f(x) - f(c)| < E \) and \( |x - c| \geq \delta \) or \( x \notin D \).

20. A function \( f \) is uniformly continuous at a set \( S \) if for every \( E > 0 \) there is a \( \delta > 0 \) such
that \(|f(x) - f(y)| < \varepsilon\) whenever \(x\) and \(y\) are in \(S\) and \(|x - y| < \delta\)

(a) \(\forall \varepsilon > 0 \exists \delta > 0\) such that \(|f(x) - f(y)| < \varepsilon \Rightarrow x, y \in S\) and \(|x - y| < \delta\)

(b) \(\exists \varepsilon > 0\) such that \(\forall \delta > 0\) such that \(|f(x) - f(y)| < \varepsilon\) and \(x, y \notin S\) or \(|x - y| \geq \delta\)

(21) The real number \(L\) is the limit of the function \(f: D \to R\) at the point \(c\) if for each \(\varepsilon > 0\) there exists a \(\delta > 0\) such that \(|f(x) - L| < \varepsilon\) whenever \(x \in D\) and \(0 < |x - c| < \delta\)

(a) \(\forall \varepsilon > 0, \exists \delta > 0\) such that \(|f(x) - L| < \varepsilon \Rightarrow x \in D\) and \(0 < |x - c| < \delta\)

(b) \(\exists \varepsilon > 0\) such that \(\forall \delta > 0\) \(|f(x) - L| < \varepsilon\) and \(x \notin D\) or \(|x - c| \geq 0\) or \(|x - c| \geq \delta\)

* Using negation to understand mathematical definitions

Part II

Find the logic connector between the sentences

(a) All men are mortal \(\forall x \in M, x \notin I\)

(b) All men are immortal \(\forall x \in M x \in I\)

(c) None of the men are mortal \(\forall x \in M, x \in I\)

(d) None of the men are immortal \(\forall x \in M, x \notin I\)

(e) There exist a man that is immortal \(\exists x \in M x \in I\)

(f) There exist a man that is mortal \(\exists x \in M x \notin I\)

\(a \iff d, b \iff c\)

Let \(C\) be the set of cats, \(c \in C\)

M(c) be the proposition "c has whiskers"

P(c) be the proposition "c likes fish"

S(c) be the proposition "c fears mice"

Write the sentence with quantifiers and the set defined

(1) The cats with whiskers always likes fish. \(\forall c \in C, M(c) \Rightarrow P(c)\)

(2) It is wrong that the cats which likes fish has whiskers. \(\exists c \in C, P(c) \land \neg M(c)\)

(3) None of the cats which like fish fear the mice. \(\forall c \in C, \neg P(c) \Rightarrow \neg S(c)\)

(4) The cats that have whiskers fear the mice. \(\forall c \in C, M(c) \Rightarrow S(c)\)

(5) The cats which fear the mice have no whiskers. \(\forall c \in C, S(c) \Rightarrow \neg M(c)\)

Suppose 1, 2, 3 is true. Make a picture presenting the set of cats. The sentences (1, 2, 3) are true. What can you say about statements 4 and 5? Statements 4 and 5 are inverses of each other.

Let \(H\) be the set of men.

We propose the two math translations for the sentence. "All men are happy and quiet."

(1) \(\forall x \in H, (x \text{ is happy } \land x \text{ is quiet})\)

All men are happy and quiet.

Is these statement the same as the first: Yes
All men are happy and all men are quiet. Is these statement the same as the first: Yes
(-) Are the 3 propositions the same: Yes

"Happy men are quiet"
(1) \( \forall x \in H, (x \text{ is happy} \implies x \text{ is quiet}) \)

Happy men are quiet
Is these statement the same as the first: Yes
(2) \( (\forall x \in H, x \text{ is happy}) \implies (\forall x \in H, x \text{ is quiet}) \)

All men are happy then all men are quiet, which has a different meaning than all men are quiet.
Is these statement the same as the first: No
(-) Are the 3 propositions the same: No

"There exist a man happy and quiet."
(1) \( \exists x \in H, (x \text{ is happy}) \land (\exists x \in H, x \text{ is quiet}) \)

There exist a man that is happy and there exist a man that is quiet.
Is these statement the same as the first: No
(2) \( \exists x \in H, [(x \text{ is happy}) \land (q \text{ is quiet})] \)

There exist a man that is happy and quiet.
Is these statement the same as the first: Yes
(-) Are the 3 propositions the same: No

"All men are not happy"
(1) \( \forall x \in H, \neg (x \text{ is happy}) \)

All men are not happy
Is these statement the same as the first: Yes
(2) \( \exists x \in H, \neg (x \text{ is not happy}) \)

There exist a man that is happy Is these statement the same as the first: No
(3) \( \neg (\forall x \in H, x \text{ is happy}) \)

There exist a man that is not happy.
Is these statement the same as the first: No
(-) Are the 3 propositions the same: No

Write the negation of the following proposition:
(a) \( p: \text{All men are kind.} \)
There exist a man that is not kind.
(b) \( p: \text{All interval of } \mathbb{R} \text{ contains an element of the interval } [0,1]. \)
There exists an interval of \( \mathbb{R} \) that does not contains an element of the interval [0,1]
(c)p:∀ p ∈ N ∀ n ∈ Z, p ≥ m
∃ p ∈ N such that ∃ n ∈ Z such that p < m
(d) (x^2 ≥ 1 ∧ x^3 < 1 ∨ (x^2 ≤ g ∧ x > 0))
(x^2 < 1 ∨ x^3 ≥ 1) ∧ (x^2 > g ∨ x ≤ 0)
(e) ∀ x ∈ R, [(x < 1) ⇒ (x^2 < 1)]
∃ x ∈ R, [(x < 1) ∧ (x^2 ≥ 1)]

Let E be a set and A, B, subsets in E. The definition of A ⊆ B is "All elements of A is an element of B"
Which of these proposition(s) below does it correspond to?
(1) ∀ x ∈ A, x ∈ B
Yes
(2) ∀ x ∈ E, (x ∈ A ⇒ x ∈ B)
Yes
(3) ∀ y ∈ E, (y ∈ A ⇒ y ∈ B)
Yes
(4) (∀ x ∈ E, x ∈ A) ⇒ (∀ x ∈ E, x ∈ B)
It is not the definition, but still is a consequence of the definition. [E ⊆ A] ⇒ [E ⊆ B]
(A ⊆ B ∧ B ⊆ A) ⇒ A=B
No

If f is a function from R to R the definition of "f is bounded" is ∃ M ∈ R, ∀ x ∈ R, |f(x)| ≤ M
Give the definition "f is not bounded"
∀ M ∈ R, ∃ x ∈ R such that |f(x)| > M

The definition of "f is increasing" is (∀ (x, y) ∈ R, x ≥ y ⇒ f(x) ≥ f(y)
Give the definition of "f is not increasing"
∃ (x, y) ∈ R such that (x ≥ y ) ∧( f(x) < f(y))

Give the definition of f is decreasing
∀ (x, y) ∈ R such that x ≥ y ⇒ f(y) ≥ f(x)

Let f be a function from {0, 1, 2} to {-1, 3, 5} defined by f(0)=3, f(1)=-1, and f(2)=5
For all these propositions give the negation and say which are true or false. (Justify your answer)
(a) ∀ i ∈ {0, 1, 2}, f(i) ≥ 0
Negation: ∃ i ∈ {0, 1, 2}, f(i) < 0
The negated statement is True so we will prove the negated statement.
Proof: Take $i=1$

$f(i) = -1 < 0$

(b) $\exists i \in \{0, 1, 2\}$, $f(i) \geq 0$

Negation: $\forall i \in \{0, 1, 2\}$, $f(i) < 0$

The negated statement is false so we will prove the original statement.

Proof: True, Take $i=0$, $f(0) = 3 \geq 0$

(c) $\forall j \in \{-1, 3, 5\}$, $\exists i \in \{0, 1, 2\}$ $f(i) = j$

Negation: $\exists j \in \{-1, 3, 5\}$ such that $\forall i \in \{0, 1, 2\}$ $f(i) \neq j$

The negated statement is false so we will prove the original statement.

Proof: Let $j \in \{-1, 3, 5\}$. If $j = -1$ then $i=1$, $f(1) = -1$, If $j = 3$ then $i=2$, $f(2) = 3$, If $j = 5$ then $i=2$, $f(2) = 5$

(d) $\forall j \in \{-1, 3\}$, $\exists i \in \{0, 1, 2\}$ $f(i) = j$

Negation: $\exists j \in \{-1, 3\}$ such that $\forall i \in \{0, 1, 2\}$ $f(i) \neq j$

Proof: Let $j \in \{-1, 3, 5\}$

If $j = -1$, Take $i=1$ therefore $f(1) = -1$
If $j = 3$, Take $i=0$ therefore $f(0) = 3$
If $j = 5$, Take $i=2$ therefore $f(2) = 5$

(e) $\forall j \in \{-1, 3, 5\}$, $\exists i \in \{0, 1, 2\}$ $f(i) = j$.

Negation: $\forall j \in \{-1, 3, 5\}$ $\exists i \in \{0, 1, 2\}$ such that $f(i) \neq j$

The negated statement is true so we will prove it.

Proof: If $j = -1$, Take $i=0$ then $f(0) \neq -1$
If $j = 3$, Take $i=2$ then $f(2) \neq 3$
If $j = 5$, Take $i=1$, then $f(1) \neq 5$

For all the propositions below. Give the negation and say if the statement is false. Justify your answer.

(a) $\forall x \in \mathbb{R}$, $x^2 > 0$

Negation: $\exists x \in \mathbb{R}$ such $x^2 \leq 0$

True, The negated statement is true so we will prove it. Take $x=0$, $x=0$ satisfies that $x \in \mathbb{R}$ and $x^2 = 0 \leq 0$

(b) $\exists x \in \mathbb{R}$, $x^2 > 0$

Negation: $\forall x \in \mathbb{R}$, $x^2 \leq 0$

False, This negated statement is false so we will prove the original statement. Take $x=2$, $x^2 = 4$, $x=2$ satisfies that $x \in \mathbb{R}$ and $x^2 = 4 > 0$

(c) $\forall x \in \mathbb{R}$, $\sqrt{x^2} = x$

Negation: $\exists x \in \mathbb{R}$ such that $\sqrt{x^2} \neq x$

True, The negated statement is true so we will prove it. Take $x=-5$, $x=-5$ satisfies that $x \in \mathbb{R}$ and $\sqrt{x^2} = 5 \neq -5$
(d) \((\forall x \in \mathbb{R}), (\exists y \in \mathbb{R}), x + y = 0\)
Negation: \((\exists x \in \mathbb{R})\) such that \((\forall y \in \mathbb{R}), x + y \neq 0\)
False, This negated statement is false so we will prove the original statement.
Let \(x \in \mathbb{R}\)
Take: \(y = -x\)
Then clearly \(x + y = 0\)

(e) \((\exists y \in \mathbb{R}), (\forall x \in \mathbb{R}), x + y = 0\)
Negation: \((\forall y \in \mathbb{R}), (\exists x \in \mathbb{R})\) such that \(x + y \neq 0\)
The negated statement is True so we will prove it.
Let \(y \in \mathbb{R}\)
Take \(x = -y + 1\)
then clearly \(x + y \neq 0\)

Explain if there is a difference between e and f and why?
Yes, there is a difference between both statements because we intertwine the quantifier, in the first statement \(y\) can be chosen in terms of \(z\) while in the second statement \(y\) is independent of \(x\).

Write with quantifiers.
(a) All natural numbers have a real square root.
\(\forall n \in \mathbb{N}, \exists x \in \mathbb{R} \text{ such that } x^2 = m\)
(b) All natural numbers have a positive real number greater than them.
\(\forall n \in \mathbb{N} \exists x \in \mathbb{R} \text{ (positive) such that } n < x\)
(c) \(\exists a \text{ real number smaller than all the integers.}\)
\(\exists x \in \mathbb{R} \text{ such that } \forall n \in \mathbb{Z} x < n\)
(d) The interval I is included in \([1,2]\)
\(\forall x \in I, x \in [1,2]\)

Let \(F\) be the of French. We denote, \(\forall \in F\),
P(x), the proposition "x has brown hats"
Q(x), the proposition "x is tall"
Answer the following questions
(1) In a picture, represent in \(F\) the set of elements of \(F\) such that P(x) is true then the set of elements such that Q is true.
(2) Consider the following sentence:
\((\forall x \in F) \ [P-[(x) \text{ or } Q(x)]]\)
and \((\forall x \in F, P(x))\) or \((\forall x \in F, Q(x))\)
Tell if the propositions are true or fall in the case of the picture.
\((\forall x \in F) \ [P(x) \text{ or } Q(x)]\)
True
\((\forall x \in F, P(x))\) or \((\forall x \in F, Q(x))\)
True
If you worked to represent when each of the sentence us true how would you represent if they are the same?

You can represent if they are the same by seeing if their pictures are equivalent or seeing if they all mean the same.

The first statement is saying All men have brown hats or are tall. While the second statement is saying all men have brown hats or all men kill. Both statements are equivalent.

Let E be a nonempty set. Denote P(x) and Q(x) as 2 propositions. Consider the following propositions.

\[ \exists x \in E, P(x) \text{ and } Q(x) \]
\[ (\exists x \in E, P(x)) \text{ and } (\exists x \in E, Q(x)) \]

Interpret the formulas P(x) and Q(x) to be pawns of the chess game. P(x) is ”x is black” and Q(x) ”x is white”

Express in english the two sentences

\[ \exists x \in E, P(x) \text{ and } Q(x) \]
There exists a pawn that is black and white

\[ (\exists x \in E, P(x)) \text{ and } (\exists x \in E, Q(x)) \]
There exist a pawn that is white and there exist a pawn that is black.

Tell me if you think the two propositions are the same or not.

These propositions are not the same. For the first proposition you are stating that there exist a pawn that is white and black simultaneously. While the second statement is stating that there are two different pawns one of which is black and the other one which is white. This condition does not have to be met simultaneously.

Write the negation of the two propositions

\[ \exists x \in E, P(x) \text{ and } Q(x) \]
\[ \text{Negation: } \neg \forall x \in E, \neg P(x) \text{ or } \neg Q(x) \]
\[ (\exists x \in E, P(x)) \text{and } (\exists x \in E, Q(x)) \]
\[ \text{Negation: } \forall x \in E, \neg P(x) \text{ or } \forall x \in E, \neg Q(x) \]

Give the logic relation between these two propositions.

\[ \exists x \in E, P(x) \text{ and } Q(x) \Rightarrow (\exists x \in E, P(x)) \text{ and } (\exists x \in E, Q(x)) \]