Problem Set # 10

Exercise 1(⋆):
Prove that for \( A, B \in M_n(K) \), we have:

1. \(||\lambda A||_\infty = |\lambda| \cdot ||A||_\infty||, \) all \( \lambda \in K \),
2. \(||A + B||_\infty \leq ||A||_\infty + ||B||_\infty||;
3. \(||AB||_\infty \leq n \cdot ||A||_\infty ||B||_\infty||.

Exercise 2(⋆):
All limits are seen in \(||·||_\infty\)-norm as \( n \to \infty \). If \( A_n \to A \) and \( B_n \to B \) and \( \lambda_n \to \lambda \) in \( \mathbb{C} \). Prove that:

1. \( A_n + B_n \to A + B \)
2. \( A_n B \to AB \) and \( AB_n \to AB \);
3. \( A_n B_n \to AB \) (matrix multiplication is jointly continuous operator in its two inputs).
   Hint: Add and subtract \( A_n B \), then apply the triangle inequality.
4. \( QA_n Q^{-1} \to QAQ^{-1} \) for \( A \) square matrix and \( Q \) invertible matrix.
5. \( \lambda_n A_n \to \lambda A \).

Exercise 3(⋆):
Determine whether the matrix
\[
A = \begin{pmatrix}
7 & -4 & 0 \\
8 & -5 & 0 \\
6 & -6 & 3
\end{pmatrix}
\]
is diagonalizable over \( K = \mathbb{R} \) or \( \mathbb{C} \). In any case, find basis vectors for eigenspace \( E_\lambda(L_A) \).

Exercise 4(⋆):
The matrix
\[
A = \begin{pmatrix}
1 & 0 & -1 & 3 \\
2 & -1 & 0 & 7 \\
1 & -2 & 3 & 5
\end{pmatrix}
\]
is row equivalent to
\[
\begin{pmatrix}
  1 & 0 & -1 & 3 \\
  0 & 1 & -2 & -1 \\
  0 & 0 & 0 & 0
\end{pmatrix}
\]

Its transpose is row equivalent to
\[
\begin{pmatrix}
  1 & 2 & 1 \\
  0 & 1 & 2 \\
  0 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}
\]

Now consider the linear map $L_A : V \rightarrow W$ between the vector spaces $V = K^4$ and $W = K^3$.

1. Find bases for $K(L_A) \subseteq V$ and $R(L_A) \subseteq W$.

2. Find bases $\mathcal{X}, \mathcal{N}$ for $V, W$ such that $[L_A]_{\mathcal{N}, \mathcal{X}}$ takes the form

\[
\begin{pmatrix}
  1 & 0 \\
  . & . & 0 \\
  0 & 1 \\
  0 & 0 & 0
\end{pmatrix}
\]

**Exercise 5(⋆):**

If the linear operator $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ has matrix

\[
A = \begin{pmatrix}
  3 & 1 & 2 \\
  1 & 0 & -1
\end{pmatrix}
\]

with respect to the standard basis, find the matrix $[T]_{\mathcal{N}, \mathcal{U}}$ of the operation with respect to the new bases

$\mathcal{U} : u_1 = (0, 1, -1), u_2 = (2, 2, -1), u_3 = (4, 0, 1)$

$\mathcal{N} : v_1 = (1, -1), v_2 = (0, -1)$