Quiz #1

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can’t make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will loose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

Problems:

1. Compute the determinant of \( A \) by cofactor expansions (choose the rows/columns wisely and start with giving the definition of determinant by cofactor expansion before computing).

\[
A = \begin{pmatrix}
6 & 0 & 0 & 5 \\
1 & 7 & 2 & -5 \\
2 & 0 & 0 & 0 \\
8 & 3 & 1 & 8 \\
\end{pmatrix}
\]

**Solution:** We expand the determinant of \( A \) across the 3rd row

\[
\begin{vmatrix}
6 & 0 & 0 & 5 \\
1 & 7 & 2 & -5 \\
2 & 0 & 0 & 0 \\
8 & 3 & 1 & 8 \\
\end{vmatrix} = a_{3,1}C_{3,1} + a_{3,2}C_{3,2} + a_{3,3}C_{3,3} + a_{3,4}C_{3,4}
\]

\[
= (-1)^{3+1} \begin{vmatrix}
6 & 0 & 5 \\
1 & 7 & -5 \\
2 & 3 & 8 \\
\end{vmatrix} = 2(-1)^{3+1} \begin{vmatrix}
7 & 2 \\
3 & 1 \\
\end{vmatrix}
\]

\[
= 10(7 - 6) = 10
\]

2. (a) Compute the determinant of \( A \) using row reduction (explain each steps of your work).

\[
A = \begin{pmatrix}
1 & 5 & -3 \\
3 & -3 & 3 \\
2 & 13 & -7 \\
\end{pmatrix}
\]
Solution:

\[
\begin{bmatrix}
1 & 5 & -3 \\
3 & -3 & 3 \\
2 & 13 & -7 \\
\end{bmatrix}
= R_2 \leftarrow R_2 - 3R_1 \text{ and } R_3 \leftarrow R_3 - 2R_1
\begin{bmatrix}
1 & 5 & -3 \\
0 & -18 & 12 \\
0 & 3 & -1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 5 & -3 \\
6 & 0 & -3 \\
0 & 3 & -1 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 5 & -3 \\
0 & -3 & 2 \\
0 & 0 & 1 \\
\end{bmatrix}
= R_3 \leftarrow R_3 + R_2
\]

(b) Give the definition of \(A^T\). Determine \(\det(A^T)\).

\textbf{Solution:} \(A^T\) is the matrix constructed from \(A\) whose rows are the columns of \(A\). We have shown in class that \(\det(A^T) = \det(A) = -18\).

(c) Decide about the linear independence of the row of \(A\). Justify your answer.

\textbf{Solution:} Since \(\det(A^T) = -18 \neq 0\), from the invertible matrix theorem we know that the row of \(A\) are linearly independent.

3. Give the definition of a linear map \(T : \mathbb{R} \to \mathbb{R}\). Determine whether the map \(T : \mathbb{R} \to \mathbb{R}\) sending \(x\) to \(x + 1\) is linear or not. (If it is give a full proof. If not give a specific precise counter example.)

\textbf{Solution:} A linear map \(T : \mathbb{R} \to \mathbb{R}\) is a map such that for all \(x, y \in \mathbb{R}\) and \(c\) scalar, such that \(T(x + y) = T(x) + T(y)\) and \(T(cx) = cT(x)\). For \(x = 1\) and \(y = -1\) and \(T(1 - 1) = T(0) = 1\) and \(T(1) + T(-1) = 2 + 0 = 2\) so that \(T(1 + (-1)) \neq T(1) + T(-1)\). So, \(T\) is not linear.