Problem Set # 4

The following equations are considered over the reals numbers.

1 Matrix Operation

Problem 1 :
Let \( A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 9 \\ -3 & k \end{pmatrix} \). What value(s) of \( k \), if any, will make \( AB = BA \)?

Problem 2 :
Let \( A = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix} \). Construct a \( 2 \times 2 \) matrix \( B \) such that \( AB \) is the zero matrix. Use two different nonzero columns of \( B \).

Problem 3 :
Let \( A = \begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} \) and \( AB = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix} \), determine the first and second columns of \( B \).

Problem 4 :
Prove that for any matrices \( m \times n \) \( A \) and \( n \times r \) \( B \),

\[
(AB)^T = B^T A^T
\]

2 Inverse

Problem 5 :

1. Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Suppose that \( \det(A) \neq 0 \). Prove that \( A \) is invertible, and that

\[
A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]
2. Let the system

\[
\begin{align*}
8x_1 + 6x_2 &= 2 \\
5x_1 + 4x_2 &= -1
\end{align*}
\]

(a) Write the matrix equation associated to the system.
(b) Write the inverse of the coefficient matrix associated to the system.
(c) Deduce from the previous question, the solution set of the system.

3 Flexibility

A horizontal elastic beam is supported at each end and is subjected to forces at points 1, 2 and 3. Let \( f \) in \( \mathbb{R}^3 \) list the forces at these points and let \( y \) in \( \mathbb{R}^3 \) list the amounts of deflection (that is, movement) of the beam at the three points. Using Hooke’s law from physics, it can be shown that

\[ y = Df \]

where \( D \) is a flexibility matrix. Its inverse is called the stiffness matrix. We can describe the physical significance of the columns of \( D \) and \( D^{-1} \), as follow.

Write \( I_3 = [e_1, e_2, e_3] \) and observe that

\[ D = DI_3 = [De_1, De_2, De_3] \]

Interpret the vector \( e_1 = (1, 0, 0) \) as a unit force applied downward at point 1 on the beam (with zero force at the other two points). Then \( De_1 \), the first column of \( D \), lists the beam deflections due to a unit force at point 1. Similar descriptions apply to the second and third columns of \( D \).

To study the stiffness matrix \( D^{-1} \), observe that the equation \( f = D^{-1}y \) computes a force vector \( f \) when a deflection vector \( y \) given. Write

\[ D^{-1} = D^{-1}I_3 = [D^{-1}e_1, D^{-1}e_2, D^{-1}e_3] \]

Now interpret \( e_1 \) as a deflection vector. Then \( D^{-1}e_1 \) lists the forces that create the deflection. That is, the first column of \( D^{-1} \) lists forces that must be applied at the three point to produce a unit deflection at point 1 and zero deflections at the order points. Similarly, columns 2 and 3 of \( D^{-1} \) list the forces required to produce unit deflections at point 2 and 3, respectively. In each column, one or two of the forces must be negative (point upward) to produce a unit deflection in the desired point and zero deflection at
the other two points. If the flexibility is measured, for example in inches of deflection per pound of load, then the stiffness matrix entries are given in pounds of load inch of deflection.

**Problem 6 :**
Let

\[
D = \begin{pmatrix}
0.011 & 0.003 & 0.001 \\
0.003 & 0.009 & 0.003 \\
0.001 & 0.003 & 0.011 \\
\end{pmatrix}
\]

Suppose that forces of 40, 50 and 30 lb are applied at points 1, 2 and 3, in Fig 1. Find the corresponding deflections.

4 Partitioned matrix

A key feature of our work with matrices has been the ability to regard a matrix \(A\) as a list of column vectors rather than just a rectangular array of numbers. This point of view has been so useful that we wish to consider other partition of \(A\), indicated by horizontal and vertical dividing rules. Partitioned matrix appear in most modern applications of linear algebra because the notation highlights essential structures in matrix analysis. Addition and scalar multiplication of partitioned matrices are quite clear. For multiplication, we use the usual Row-Column rule as if the block entries were scalars provided that for a product \(AB\), the column partition of \(A\) matches the row partition of \(B\). Then we say that the partition of \(A\) and \(B\) are conformable for block multiplication. **Problem 7 :**

The matrix

\[
A = \begin{pmatrix}
2 & 0 & -4 & 3 \\
6 & 0 & -12 & 4 \\
0 & 1 & 0 & 5 \\
\end{pmatrix}
\]

can be also written as

\[
A = \begin{pmatrix}
A_{1,1} & A_{1,2} \\
A_{2,1} & A_{2,2} \\
\end{pmatrix}
\]

where

\[
A_{1,1} = \begin{pmatrix}
2 & 0 \\
6 & 0 \\
\end{pmatrix}
\]

\[
A_{1,2} = \begin{pmatrix}
-4 & 3 \\
-12 & 4 \\
\end{pmatrix}
\]

\[
A_{2,1} = \begin{pmatrix}
0 & 1 \\
\end{pmatrix}
\]

\[
A_{2,2} = \begin{pmatrix}
0 & 5 \\
\end{pmatrix}
\]

Similarly, if

\[
B = \begin{pmatrix}
1 & 0 \\
6 & 0 \\
0 & 5 \\
1 & 0 \\
\end{pmatrix}
\]
Then

\[ B = \begin{pmatrix} B_{1,1} \\ B_{2,1} \end{pmatrix} \]

with

\[ B_{1,1} = \begin{pmatrix} 1 & 0 \\ 6 & 0 \end{pmatrix} \]

and

\[ B_{2,2} = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix} \]

The partitioned matrices are conformable for block multiplication. Then

\[ AB = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} B_{1,1} \\ B_{2,1} \end{pmatrix} = \begin{pmatrix} A_{1,1}B_{1,1} + A_{1,2}B_{2,1} \\ A_{2,1}B_{1,1} + A_{2,2}B_{2,1} \end{pmatrix} \]

BE CAREFUL THE ORDER MATTERS since the matrix multiplication is not commutative.

Compute \( AB \) using this formula and compute \( AB \) using the usual matrix multiplication without using the partition of \( A \) or \( B \).

A block diagonal matrix is a partitioned matrix with zero blocks off the main diagonal (of blocks). Such a matrix is invertible if and only if each block on the diagonal is invertible.

**Problem 8 :**
A matrix of the form

\[ A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ 0 & A_{2,2} \end{pmatrix} \]

is said to be block upper triangular. Assume that \( A_{1,1} \) is \( p \times p \) and \( A_{2,2} \) is \( q \times q \) and are invertible. Find a formula for \( A^{-1} \).