1 The matrix equation $Ax = b$

The following equations are considered over the reals numbers.

**Problem 1:**

Let $u = \begin{pmatrix} 0 \\ 4 \\ 4 \end{pmatrix}$ and $A = \begin{pmatrix} 3 & 5 \\ -2 & 6 \\ 1 & 1 \end{pmatrix}$. Is $u$ in the plane passing through the origin with directions the columns of $A$?

**Problem 2:**

Let

$$A = \begin{pmatrix} 4 & -5 & -1 & 8 \\ 3 & -7 & -4 & 2 \\ 5 & -6 & -1 & 4 \\ 9 & 1 & 10 & 7 \end{pmatrix}$$

Determine if the columns of the matrix $A$ span $\mathbb{R}^4$.

2 Solution sets of linear system

**Problem 3:**

1. Write the solution set of the given homogeneous system in parametric vector form, express it as a Span and give its geometric description,

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 + x_2 - 3x = 0 \\ -x_1 + x_2 = 0 \end{cases}$$

2. Describe the solutions of the following system in parametric vector form, and provide a geometric comparison with the solution set of the previous question.

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 5 \\ 2x_1 + x_2 - 3x_3 = 13 \\ -x_1 + x_2 = -8 \end{cases}$$
3 Application of linear system

Problem 4 :
Chemical equations describe the quantities of substances consumed and produced by chemical reactions. Alka-Selter contains sodium bicarbonate \((NaHCO_3)\) and citric acid \((H_3C_6H_5O_7)\). When a tablet is dissolved in water the following reaction produces sodium citrate, water and carbon dioxide (gas):

\[
NaHCO_3 + H_3C_6H_5O_7 \rightarrow Na_3C_6H_5O_7 + H_2O + CO_2
\]

Balance the chemical equation. (For this, a systematic method for balancing chemical equation is to set up a vector equation that describes the numbers of atoms of each type present in a reaction. Since the equation involves 4 atoms: Sodium (Na), Hydrogen (H), carbon (C) and oxygen (O). For instance, \(NaHCO_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}\). Then translate the problem into a vector equation.)

Problem 5 :
A network consist of a set of points called junction or nodes, with lines or arcs, called branches connecting some or all of the junction. The direction of flow of each branch is indicated and the flow amount (or rate) is either shown or is denoted by a variable. The basic assumption of network flow is that the total flow into the network equals the total flow out of the junction. For instance

\[
\begin{align*}
40 & \quad \rightarrow \\
\uparrow & \\
x_1 & \\
\downarrow & \\
\rightarrow & \\
\downarrow & \\
x_2
\end{align*}
\]

Shows that 40 units flowing into a junction through one branch, with \(x_1\) and \(x_2\) denoting the flows out of the junction through other branches. Since the flow is conserved at each junction, we must have \(x_1 + x_2 = 40\). In a similar fashion, the flow at each junction is described by a linear equation. The problem of network analysis is to determine the flow in each branch when partial information (such as the flow into and out the network) is known.

Find the general flow pattern of the network shown in the figure. Assuming the flows
are all nonnegative, what is the smallest possible value for \( x_4 \)?

\[
\begin{array}{c}
A \\
B \\
C
\end{array}
\quad
\begin{array}{c}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{array}
\quad
\begin{array}{c}
100 \\
80
\end{array}
\]

(For this complete the table)

<table>
<thead>
<tr>
<th>Intersection</th>
<th>Flow in</th>
<th>Flow out</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
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<tr>
<td>B</td>
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<tr>
<td>C</td>
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and then translate the problem into solving a system of linear equations.

Problem 6:
Suppose an economy has four sectors: Mining, Lumber, Energy, and Transportation. Mining sells 10% of its output to Lumber, 60% to Energy, and retains the rest. Lumber sells 15% of its output to Mining, 50% to Energy, 20% to Transportation, and retains the rest. Energy sells 20% of its output to Mining, 15% to Lumber, 20% to Transportation, and retains the rest. Transportation sells 20% of its output to mining, 10% to lumber, 50% to Energy, and retains the rest.

1. Construct the exchange table for this economy.
2. Find a set of equilibrium prices for this economy.