Problem Set #11

The following equations are considered over the reals numbers. All the answers should be justified unless mentioned differently.

Problem 1:
Let $W$ be a subspace of $\mathbb{R}^n$, and let $W^\perp$ be the set of all vectors orthogonal to $W$. Show $W^\perp$ is a subspace of $\mathbb{R}^n$.

Solution: By definition, $W^T$ is a subset of $\mathbb{R}^n$.
Let $u, v \in W^T$ and $c$ scalar, we need to prove that $u + v$ is in $W^T$ and $cu$ is in $W^T$.
We know that $<u, w> = 0$ for all $w \in W$ and $<v, w> = 0$ for all $w \in W$.
Let $w \in W$. Then
\[ <u + v, w> = <u, w> + <v, w> = 0 + 0 = 0 \]
so that $u + v \in W^T$ and
\[ <cu, w> = c <u, w> = c \cdot 0 = 0 \]
so that $cu \in W^T$.

Problem 2:
Show that if $x$ is in both $W$ and $W^\perp$, then $x = 0$.
Solution: Let $x$ in both $W$ and $W^\perp$, then $<x, x> = 0$ thus $x = 0$.

Problem 3:
Determine which sets of vectors are orthogonal. If a set is orthogonal, normalize the vector to produce an orthonormal set.

1. \[ \left( \begin{array}{c} -0.6 \\ 0.8 \end{array} \right), \left( \begin{array}{c} 0.8 \\ 0.6 \end{array} \right) \]

2. \[ \left( \begin{array}{c} 1/\sqrt{10} \\ 3/\sqrt{20} \end{array} \right), \left( \begin{array}{c} 3/\sqrt{10} \\ -1/\sqrt{20} \end{array} \right), \left( \begin{array}{c} 0 \\ -1/\sqrt{2} \end{array} \right), \left( \begin{array}{c} 0 \\ 1/\sqrt{2} \end{array} \right) \]

Solution:
1. Since $u \cdot v = 0$, $\{u, v\}$ is an orthogonal set. However, $||u||^2 = u \cdot u = 1$ and $||v||^2 = v \cdot v = 1$, so $\{u, v\}$ is an orthonormal set.

2. Also, $u \cdot v = u \cdot w = v \cdot w = 0$, $\{u, v, w\}$ is an orthogonal set. Also, $||u||^2 = u \cdot u = 1$, $||v||^2 = v \cdot v = 1$ and $||w||^2 = w \cdot w = 1$, so $\{u, v, w\}$ is an orthonormal set.
Problem 4:
Given $u \neq 0$ in $\mathbb{R}^n$, let $L = \text{Span}\{u\}$. Show that the mapping $x \mapsto \text{proj}_L(x)$ is a linear transformation.

Solution: Let $L = \text{Span}\{u\}$, where $u$ is nonzero and let $T(x) = \frac{x \cdot u}{u \cdot u} u$. For any $x, y \in \mathbb{R}^n$ and $c, d$ scalar, the algebraic properties of the inner product show that

$$T(cx + dy) = \frac{(cx + dy) \cdot u}{u \cdot u} u = \frac{c}{u \cdot u} x \cdot u + \frac{d}{u \cdot u} y \cdot u = cT(x) + dT(y)$$

Thus, $T$ is a linear transformation. Another approach is to view $T$ as the composition of the following three linear mappings: $x \mapsto x \cdot v = a$, $a \mapsto b = a/v$, and $b \mapsto bv$.

Problem 5:
Express $x$ as a linear combination of the $u$'s.

$$u_1 = \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 6 \\ 4 \end{pmatrix}, \quad \text{and} \quad x = \begin{pmatrix} 9 \\ -7 \end{pmatrix}$$

Solution: Since $u_1 \cdot u_1 = 12 - 12 = 0$. $\{u_1, u_2\}$ is an orthogonal set. Since the vectors are non-zero vectors are non-zero, $u_1$ and $u_2$ are linearly independent. Two such vectors in $\mathbb{R}^2$ automatically form a basis for $\mathbb{R}^2$. So $\{u_1, u_2\}$ is orthogonal basis for $\mathbb{R}^2$. And we know that

$$x = \frac{x \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{x \cdot u_2}{u_2 \cdot u_2} u_2 = 3u_1 + 1/2u_2$$

Problem 6:
Find the closest point to $y$ in the subspace $W$ spanned by $v_1$ and $v_2$.

$$y = \begin{pmatrix} 3 \\ 1 \\ 5 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 3 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Solution: Note that $v_1$ and $v_2$ are orthogonal. The best approximation theorem says that $\hat{z}$, which is the orthogonal projection of $z$ onto $W = \text{Span}\{v_1, v_2\}$ is the closest point to $z$ of $W$. This vector is

$$\hat{y} = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = 1/2v_1 + 3/2v_2 = \begin{pmatrix} 3 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

Problem 7:
Find the best approximation to $z$ by vectors of the form $c_1v_1 + c_2v_2$.

$$z = \begin{pmatrix} 3 \\ -7 \\ 2 \\ 3 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 2 \\ -1 \\ -3 \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$
Solution : Note that $v_1$ and $v_2$ are orthogonal. The best approximation theorem says that $\hat{y}$, which is the orthogonal projection of $y$ onto $W = \text{Span}\{v_1, v_2\}$ is the closest point to $y$ of $W$. This vector is 

$$\hat{z} = \frac{z \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{z \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{pmatrix} -1 \\ -3 \\ -2 \\ 3 \end{pmatrix}$$

Problem 8 :

$$A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix}$$

1. Find an orthogonal basis for $\text{Col}(A)$.
2. Find a $QR$ factorization of $A$.

Solution :

1. Call the columns of the matrix $x_1$, $x_2$ and $x_3$ and perform the Gram-Schmidt process on these vectors :

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = x_2 - (-1) v_1 = \begin{pmatrix} 3 \\ 0 \\ 3 \\ -3 \end{pmatrix}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = x_3 - 4v_1 - (-1/3)v_2 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ -2 \end{pmatrix}$$

Thus an orthogonal basis for $W$ is 

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 3 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ -2 \end{pmatrix} \right\}$$

2. The columns of $Q$ will be normalized versions of the vectors $v_1, v_2$ and $v_3$ found in the previous question.
Thus

\[
Q = \begin{pmatrix}
1/\sqrt{5} & 1/2 & 1/2 \\
-1/\sqrt{5} & 0 & 0 \\
-1/\sqrt{5} & 1/2 & 1/2 \\
1/\sqrt{5} & -1/2 & 1/2 \\
1/\sqrt{5} & 1/2 & -1/2
\end{pmatrix}
\]

and since \( A = QR \) and \( Q \) has orthonormal columns:

\[
R = Q^T A = \begin{pmatrix}
\sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\
0 & 6 & -2 \\
0 & 0 & 4
\end{pmatrix}
\]