Problem Set # 8

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can’t make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will lose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

Exercise 1:
Find the Jordan canonical form for the linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$ whose matrices with respect to the standard basis $\mathcal{X} = \{e_1, \ldots, e_n\}$ are

1.

$$A = \begin{pmatrix}
7 & 1 & 2 & 2 \\
1 & 4 & -1 & -1 \\
-2 & 1 & 5 & -1 \\
1 & 1 & 2 & 8
\end{pmatrix}$$

2.

$$B = \begin{pmatrix}
5 & -6 & -6 \\
-1 & 4 & 2 \\
3 & -6 & -4
\end{pmatrix}$$

Solutions:

1. see class notes.

2. You can compute the characteristic polynomial

$$p_A(x) = \text{det}(A - xI) = -x^3 + 5x^2 - 8x + 4$$

We can find one root by trial and error. In fact $p_A(0) = 4$, $p_A(1) = -1+5-8+4 = 0$. So $\lambda = 1$ is an eigenvalue. Long division by $x-1$ yields a complete factorization of $p_A$.

$$p_A(x) = -(x-1)(x^2 - 4x + 4) = -(x-1)(x-2)^2$$

So $Sp_C(A) = \{1, 2\}$.  

1
(a) **Case 1: \( \lambda = 1 \)** Find \( K_1(A-I) \): solve \( (A-I)x = 0 \).

\[
\begin{pmatrix}
4 & -6 & -6 \\
-1 & 3 & 2 \\
3 & -6 & 5
\end{pmatrix} \sim \begin{pmatrix}
1 & -3 & -2 \\
0 & 3 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

so \( x_3 \) = free variable. \( x_2 = -1/3x_3 \), \( x_1 = 3x_2 + 2x_3 = -x_3 + 2x_3 = x_3 \). Thus,

\[
K_1(A-I) = E_{\lambda=1} = \mathbb{C} \cdot \begin{pmatrix} 1 \\ -1/3 \\ 1 \end{pmatrix} = \mathbb{C} \cdot \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix}
\]

Recall \( \dim(M_\lambda) = \text{alg} \cdot \text{mult} \) of \( \lambda \); \( \dim(E_\lambda) = \text{geometric multiplicity} \) of \( \lambda \).

Since \( \text{alg} \cdot \text{mult}(\lambda = 1) = 1 \) no further calculations is needed to see that \( M_{\lambda=1} = E_{\lambda=1} \) is 1-dimensional.

(b) **Case 2: \( \lambda = 2 \)** Solve \( (A-2I)x = 0 \) via row operations:

\[
A-2I = \begin{pmatrix}
3 & -6 & -6 \\
-1 & 2 & 2 \\
3 & -6 & -6
\end{pmatrix} \sim \begin{pmatrix}
1 & -2 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

So \( x_2, x_3 \) are free variable; and \( x_1 = 2x_2 + 2x_3 \), and

\[
K_1(A-2I) = E_{\lambda=2} = \left\{ \begin{pmatrix} 2x_2 + 2x_3 \\ x_2 \\ x_3 \end{pmatrix} : x_2, x_3 \in \mathbb{C} \right\} = \mathbb{C} \cdot \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \oplus \mathbb{C} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

has dimension 2. Recall that the spam \( W = E_{\lambda=1} + E_{\lambda=2} \) has dimension \( |W| = |E_{\lambda=1}| + |E_{\lambda=2}| = |\mathbb{C}^3| \), hence \( \mathbb{C}^3 = E_{\lambda=1} \oplus E_{\lambda=2} \). The operator is diagonalizable with diagonalizing basis

\[
\mathcal{Y} = \{ f_1 = (3,-1,3); f_2 = (2,1,0); f_3 = (2,0,1) \}
\]

and

\[
[L_A]_{\mathcal{Y}, \mathcal{Y}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}
\]

Since \( E_\lambda \subseteq M_\lambda \), for every eigenvalue and \( V = \oplus_{\lambda \in \text{sp}(A)} M_\lambda \) and since \( E_\lambda \subseteq M_\lambda \), the fact that \( E_{\lambda=1} \oplus E_{\lambda=2} \) already have dimension 3 implies that \( M_{\lambda=1} = E_{\lambda=1} \) and \( M_{\lambda=2} = E_{\lambda=2} \), and no further computations are needed to determine the generalized eigenspaces for \( L_A \).

**Exercise 2:** From the Canonical Jordan form theorem seen in class, deduce that if \( A \) is \( n \times n \) matrix whose characteristic polynomial \( p_A(t) = \text{det}(A - tI) \) splits over \( \mathbb{F} \), \( A \) is diagonalizable if and only if \( \text{geometric \cdot multiplicity}(\lambda) = \text{algebraic \cdot multiplicity}(\lambda) \) for each \( \lambda \in \text{sp}(A) \).
Solution: See class notes.

Exercise 3: Suppose $T : V \to V$ has characteristic polynomial $p_T(t) = (-1)^n t^n$.

1. Are all such operator nilpotent? Prove or give a counterexample.

2. Does the nature of the ground field $\mathbb{F}$ matter in answering this question?

Solution: We know if $T$ is nilpotent then $p_T(t) = (-1)^n t^n$ because if $V = V_1 \oplus \cdots \oplus V_r$, cyclic subspaces) then

$$p_T(t) = det(T - tI) = \prod_{i=1}^{r} det(T - tI)|_{V_i} = \prod_{i=1}^{r} det(T|_{V_i} - tI_{V_i}) = \prod_{i=1}^{r} p_{T|_{V_i}}(t)$$

But for a cyclic base $X_i$ in $V_i$, we get

$$[T|_{V_i}]_{X_i} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots \\ 0 & 1 \\ 0 \end{pmatrix}$$

and

$$P_{T|_{V_i}}(t) = det \begin{pmatrix} (-t) & 1 & 0 & \cdots & 0 \\ 0 & (-t) & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & (-t) \end{pmatrix} = (-1)^{d_i} t^{d_i}, \ (d_i = \dim_{\mathbb{F}} V_i)$$

Thus, $p_T(t) = \prod_{i=1}^{r} (-1)^{d_i} t^{d_i} = (-1)^n t^n$ where $n = \dim(V)$.

Now suppose we only know that $p_T(t) = (-1)^d t^n$. The only eigenvalue is $\lambda = 0$ and $p_T$ splits into linear factors in $\mathbb{F}[t]$ for any ground field $\mathbb{F}$. By results developed in class notes, $\dim(M_{\lambda=0}(T)) = Alg \cdot \text{multiplicity of } \lambda = 0$ on $p_T(t)$, with is obviously $n$.

Hence, $M_{\lambda=0}(T) = \text{all of } V$. We have previously shown that for any $T : V \to V$ such that $p_T$ splits over $\mathbb{F}$, $(T - \lambda I)|_{M_{\lambda}}$ is nilpotent (because for a suitable basis,

$$[T]_{X} = \begin{pmatrix} A_1 & 0 & 0 \\ 0 & A_2 & \cdots \\ \vdots & \vdots & \ddots \\ 0 & 0 & A_r \end{pmatrix}$$
where
\[
A_i = \begin{pmatrix}
\lambda & 1 & 0 \\
0 & \lambda & 1 \\
& & \ddots & \ddots \\
& & & \lambda & 1 \\
& & & & \lambda
\end{pmatrix}
\]
and
\[
[T - \lambda I]_\mathcal{X} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
& & & \ddots \\
& & & & 0 & 1 \\
& & & & & 0
\end{pmatrix}
\]
is obviously nilpotent.) Thus \(T\) is nilpotent over \(\mathcal{F}\) if and only if \(p_T(t) = (-1)^n t^n\), for any field \(\mathbb{F}\).

**Exercise 4:** Let \(T : V \to V\) be an \(\mathbb{F}\)-linear operator on a finite dimensional vector space such that the characteristic polynomial \(p_T(t)\) splits over \(\mathbb{F}\). Let \(\mathcal{X}\) be a basis such that \(J = [T]_\mathcal{X}\) has Jordan canonical form, and let \(J^T\) be the transpose of this matrix.

1. Explain why \(J^T\) and \(J\) are similar matrices.

2. If \(A\) is any matrix in \(M(n, \mathbb{F})\) whose characteristic polynomial splits over \(\mathbb{F}\), prove that the transpose \(A^T\) is always related to \(A\) by similarity transformation.

**Note:** In particular, 2. holds for every matrix \(A\) if \(\mathbb{F} = \mathbb{C}\).

**Solution:** Let \(T = L_A : V \to V\) where \(V = \mathbb{F}^n\), \(L_A(v) = A \cdot \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}\) (matrix product). Then \(A = [T]_\mathcal{X}\) if \(\mathcal{X} = \{e_1, \ldots, e_n\}\) is the standard basis in \(V = \mathbb{F}^n\). We know \(V = \oplus_{\lambda_i \in \text{Sp}(T)} M_{\lambda_i}\) and that in each space \(M_{\lambda_i}\) there is a basis \(\mathcal{Y}_i\) such that \(T_i = T|_{M_{\lambda_i}}\) is such that \([T_i]_{\mathcal{Y}_i} = \begin{pmatrix} J_1 & 0 \\ & \ddots \\ & & J_m \end{pmatrix}\) with
\[
J_j = \begin{pmatrix}
\lambda_i & 1 & 0 \\
& & \ddots & \ddots \\
& & & \lambda_i
\end{pmatrix}
\]
where the blocks correspond to a decomposition $M_{\lambda_i} = W_1 \oplus \cdots \oplus W_s$ into cyclic subspaces under the nilpotent operator $T - \lambda_i I$ on $M_{\lambda_i}$. The basis $\mathcal{Y}_i$ in $M_{\lambda_i}$ can be written as $\mathcal{Y}_i = \{\mathcal{Y}^{(i)}_1, \cdots, \mathcal{Y}^{(i)}_s\}$ where $\mathcal{Y}^{(i)}_k$ is the cyclic basis for $(T - \lambda_i I)$ in $W_k$, $1 \leq k \leq s$.

If we reverse the order of the vectors in each $S$ space $W_k$, the resulting matrix for $T|_{W_k}$ is the transpose of that

$$[T_i]_{\mathcal{Y}'} = \begin{pmatrix}
    J^T_{f_1} & 0 \\
    \vdots & \ddots \\
    J^T_{f_m} & 
\end{pmatrix}
$$

with

$$J^T_{f_j} = \begin{pmatrix}
    \lambda_i & 0 \\
    1 & \ddots \\
    \vdots & \ddots & \ddots \\
    0 & \cdots & 1 & \lambda_i 
\end{pmatrix}
$$

In fact, if $\mathcal{Y}^{(i)}_k = \{f_1, \cdots, f_m\}$, we must have

$$f_m \rightarrow^{T-\lambda_i I} f_{m-1} \rightarrow^{T-\lambda_i I} \cdots \rightarrow^{T-\lambda_i I} 0
$$

hence if we reverse the order to get $\mathcal{Y}^{(i')}_{k'} = \{f'_1, \cdots, f'_m\} = \{f_m, \cdots, f_1\}$, we have

$$f'_1 \rightarrow^{T-\lambda_i I} f'_2 \rightarrow^{T-\lambda_i I} \cdots f'_{m-1} \rightarrow^{T-\lambda_i I} f'_m \rightarrow^{T-\lambda_i I} 0
$$

and then the $k^{th}$ block in $[T|_{M_{\lambda_i}}]_{\mathcal{Y}'}$ is $[T|_{W_k}]_{\mathcal{Y}^{(i')}_{k'}}$ the $k^{th}$ blovk in $[T_i]_{\mathcal{Y}_i}$. Then there is $S_2$ such that $S_2A_2S_2^{-1} = A_3$ consists of blocks of the form show in $[T_i]_{\mathcal{Y}_i}$. Here,

$$A_3 = A_2^T = (S_1A_1S_1^{-1})^T = (S_1^T)^{-1}A_1^T(S_1^T)
$$

Therefore

$$A_1^T = (S_1^T)^{-1}A_3(S_1^T) = (S_1^T)^{-1}S_2A_2S_2^{-1}(S_1^T) = (S_1^T)^{-1}S_2S_1A_1S_1^{-1}S_2^{-1}(S_1^T) = SA_1S^{-1}
$$

As claimed.

**Exercise 5:**
If $T : V \rightarrow V$ is an arbitrary linear operator over $\mathbb{C}$ explain how the components of the Fitting decomposition $V = K_\infty \oplus R_\infty$ are related to the generalized eigenspace decomposition $V = \bigoplus_{\lambda \in \text{sp}(T)} M_\lambda(T)$.

**Solution:** $K_\infty$ in $V = K_\infty \oplus R_\infty$ consist of all vectors of $T^i(v) = 0$ eventually as $i \rightarrow \infty$. This is precisely the generalized eigenspace $M_{\lambda=0} = \{v : (T - \lambda I)(v) = 0 : \text{for some } i\}$ (taking $\lambda = 0$). The stable range of $T$ is actually $R_\infty = \bigoplus_{\lambda \neq 0, \lambda \in \text{sp}(T)} M_\lambda$ in block
upper triangular form \((Sp(T) = \lambda_0, \ldots, \lambda_r \text{ with } \lambda_0 = 0)\) This operator leaves invariant \(W = \bigoplus_{\lambda \neq 0} M_\lambda\) and the restriction \(T|_W\) has upper \(\lambda\) matrix with diagonal values \(\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_r, \ldots, \lambda_r\); thus \(\text{det}(T|_W) \neq 0\). So there is a basis such that the matrix of \(T\) is of the form:

\[
\begin{pmatrix}
B_1 & 0 \\
\vdots & \ddots \\
0 & B_r
\end{pmatrix}
\]

with

\[
B_i = \begin{pmatrix}
\lambda_i & * \\
\vdots & \ddots \\
0 & \lambda_i
\end{pmatrix}
\]

Obviously, \(W \subseteq R_\infty\).

But

\[|M_{\lambda=0}| + |W| = |V| = |K_\infty| + |R_\infty|\]

and \(|K_\infty| = |M_{\lambda=0}|\). Thus \(|R_\infty| = |W|\) and \(R_\infty = W\) as subspaces. Therefore,

\[V = R_\infty \oplus R_\infty \text{ is } \left(\bigoplus_{\lambda \neq 0, \lambda \in Sp(T)} M_\lambda\right) \oplus M_{\lambda=0}\]

**Note:** If \(\lambda_0\) is any complex number in \(Sp_C(T)\), the fitting decomposition of \(T - \lambda_0\) is \(\left(\bigoplus_{\lambda \neq \lambda_0, \lambda \in Sp(T)} M_\lambda\right) \oplus M_{\lambda_0}\), by the same argument. (But now \(K_\infty = K_\infty(T - \lambda_0)\) and \(R_\infty = R_\infty(T - \lambda_0))\).

**Exercise 6:**

Use the Jordan Canonical decomposition to show that all solutions \(A \in M(n, \mathbb{C})\) to the matrix equation \(A^2 + I = 0\) are similar to matrices of the form

\[B = \begin{pmatrix}
iI_r & 0 \\
0 & -iI_s
\end{pmatrix}\]

where \(r + s = n\) and \(i = \sqrt{-1}\).

**Solution:** First Show \(sp_C(A) \subseteq \{\pm i\}\), i.e. the only possible eigenvalues. In fact \(Av = \lambda v\ (v \neq 0) \Rightarrow A^2v = \lambda^2v\) and if \(A^2 = -I\) that means \(\lambda^2v = A^2v = -Iv = -v \Rightarrow (\lambda^2 + 1) = 0\) since \(v \neq 0 \Rightarrow \lambda = \pm i\).

Then by Jordan decomposition \(\exists S \in GL(n, \mathbb{C})\) such that

\[
SAS^{-1} = \begin{pmatrix}
B_1 & 0 \\
\vdots & \ddots \\
0 & B_r
\end{pmatrix}
\]
with

\[ B_i = \begin{pmatrix} \lambda_i & * \\ \cdot & \cdot \\ 0 & \lambda_i \end{pmatrix}_{d_i \times d_i} \]

where \( \lambda_k = \pm i \).

Then

\[ SA^2S^{-1} = \begin{pmatrix} B_1^2 & 0 \\ \cdot & \cdot \\ 0 & B_r^2 \end{pmatrix} \]

So, \( B_k^2 = -Id_{d_k \times d_k} \) for each \( k \).

Now, write \( B_k = \lambda_k I + N_k \) (\( N_k = \begin{pmatrix} 0 & 1 & 0 \\ \cdot & \cdot & 1 \\ 0 & 0 & 0 \end{pmatrix} \)). Then

\[ B_k = \lambda_k^2 I + 2\lambda_k N_k + N_k^2 = \begin{pmatrix} \lambda_k^2 & 2\lambda_k & 1 & 0 \\ \cdot & \cdot & 1 & \cdot \\ 0 & \cdot & 2\lambda_k & \cdot \\ \cdot & \cdot & \cdot & \lambda_k^2 \end{pmatrix} \]

Since

\[ N^2 = \begin{pmatrix} 0 & 1 & 0 \\ \cdot & \cdot & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 \\ \cdot & \cdot & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

Therefore

\[ B_k^2 = -Id_{d_k \times d_k} = \begin{pmatrix} -1 & 0 \\ \cdot & \cdot \\ 0 & -1 \end{pmatrix} \]

This is impossible due to off diagonal terms in \( B_k^2 \) unless \( d_k = 1 \), so \( B_k \) is \( 1 \times 1 \) and of form \( B_k = [\lambda_k] \) with \( \lambda_k = \pm i \). Thus \( SAS^{-1} \) is diagonal:

\[ SAS^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ \cdot & \cdot \\ 0 & \lambda_n \end{pmatrix} \]
\( \lambda_k = \pm i \). Relabeling basis eigenvectors, there is another \( Q \in GL(n, \mathbb{F}) \) such that

\[
Q \begin{pmatrix}
\lambda_1 & 0 \\
\vdots & \ddots & 0 \\
0 & \cdots & \lambda_n
\end{pmatrix} Q^{-1} = \begin{pmatrix}
iI_r & 0 \\
0 & -iI_s
\end{pmatrix}
\]

where \( r, s \geq 0 \) and \( r + s = n \). Thus, \( A \) is similar to

\[
QSA(QS)^{-1} = \begin{pmatrix}
iI_r & 0 \\
0 & -iI_s
\end{pmatrix}
\]

as claimed.