Problem Set # 7

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can’t make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will lose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

Exercise 1:
Prove that the Fitting decomposition is unique: If \( V = N \oplus S \), both \( T \)-invariant, such that \( T|_N \) is nilpotent and \( T|_S : S \to S \) invertible show that \( N = K_\infty(T) \) and \( S = R_\infty(T) \).

Exercise 2: Let \( T : V \to V \) be a linear operator on some finite dimensional vector space over an arbitrary field \( \mathbb{F} \). Suppose \( M_\lambda \) is the generalized \( \lambda \)-eigenspace for some \( \lambda \in \text{Sp}_\mathbb{F}(T) \) and let \( M \subseteq M_\lambda \) be some subspace that is invariant under \( (T - \lambda) \) \( \Leftrightarrow \) it is invariant under \( T \). Assume \( M \) is cyclic i.e there exists a vector \( e_0 \in M \) such that \( e_j = (T - \lambda)^j(e_0) \) span \( M \).

(a) Verify that \( (T - \lambda) \) is nilpotent on both \( M_\lambda \) and \( M \).

If \( m \geq 1 \) is the largest exponent such that the vectors
\[
e_0, (T - \lambda)e_0, \ldots, (T - \lambda)^{m-1}e_0 \quad (*)
\]
are non-zero (so \( (T - \lambda)^m e_0 = 0 \)).

(b) Prove that these vectors are linearly independent and form a basis for \( M \).

1. If you list these basis vectors in the order shown in (*) what is the matrix form of the restricted operator \( T|_M \)?

(c) If \( \mathcal{N} \) is the basis obtained by listing the vectors in (*) in reverse order, show that \( [T|_M]_{\mathcal{N}} = [T|_M]_{\mathcal{X}}^T \) (transpose matrix).

2. Let \( T \) be any linear operator and \( \mathcal{X} \) and ordered basis. Let \( \mathcal{N} \) be the basis obtained by reversing the order of the vectors in \( \mathcal{X} \). It is always true that \( [T]_{\mathcal{N}} = [T]_{\mathcal{X}}^T \)?
Exercise 3: Verify that the matrix

\[
A = \begin{pmatrix}
-3 & 1 & 2 \\
-3 & -1 & 1 \\
-6 & 2 & 4 \\
\end{pmatrix}
\]

is nilpotent. Find a basis \( \mathcal{X} \) in \( \mathbb{F}^3 \) that gives a cyclic subspace decomposition and compute the matrix of \( L_A : \mathbb{F}^3 \to \mathbb{F}^3 \) with respect to this basis. What are the generalized eigenspaces for \( L_A \)?

Exercise 4: Let \( V = \mathcal{P}_n \) be all polynomials in \( \mathbb{R}[x] \) of degree \( \leq n \). Show that the "differentiation operator" \( D : \mathcal{P}_n \to \mathcal{P}_n \)

\[
D(a_0 + a_1 x + \cdots + a_n x^n) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1}
\]

is nilpotent of degree \( n + 1 \). Find a basis that puts \( D \) into Jordan form.

Exercise 5:

Repeat the analysis of the example 4.14 of the notes for the matrix \( A = \begin{pmatrix} 4 & 4 \\ -1 & 0 \end{pmatrix} \).