Problem Set # 5

Exercise 1:
Explain why every linear operator $T : V \to V$ on a finite dimensional vector space over $\mathbb{C}$ always has an eigenvalue.

Exercise 2:
Does the concept of “self adjoint make sense for linear operators $T : V \to V$ on a finite dimensional vector space over $\mathbb{C}$ if the space $V$ does not come equipped with an inner product? Here’s what we mean: given $T : V \to V$, suppose there is a basis $X$ such that $A = [T]_X$ is a self-adjoint matrix, in the sense that $A^* = A$. If $\mathcal{N}$ is any other basis does the matrix $B = [T]_N$ always have this property? Prove or provide a counter example.
Note: If the answer is no, then “self-adjointness” is not a coordinate independent property of linear operators on complex vector spaces. An adjoint operator $T^*$ can only be defined when $V$ is equipped with the extra structure of an inner product.

Exercise 3:
Let $W$ be the linear span of three independent vectors $\{v_1, v_2, v_3\}$ in a finite dimensional vector space $V$.

1. If $V = \mathbb{C}^5$ and $v_1 = (1, 1, 2, 0, 3)$, $v_2 = (3, 2, 1, 5, -1)$, $v_3 = (2, 1, 0, 2, -1)$. Find an explicit choice of 2 vectors $e_{i_1}, e_{i_2}$ from the standard basis $\{e_1, \cdots , e_5\}$ such that $\{v_1, v_2, v_3, e_{i_1}, e_{i_2}\}$ is a basis for $\mathbb{C}^5$.

2. If $\{f_1, \cdots , f_n\}$ is a particular basis in $V$, prove that one can always create a basis for $V$ by augmenting the $v_i$ with $n - 3$ vectors $f_{i_1}, \cdots , f_{i_{n-3}}$ selected from the given basis.

3. From 2., explain why the cosets $\{\overline{f_{i_1}}, \cdots , \overline{f_{i_{n-3}}}\}$ in the quotient space $V/W$ are a basis for $V/W$.

Exercise 4: Let $T : V \to V$ be an arbitrary linear map and $W$ a $T$-invariant subspace. We say that vectors $e_1, \cdots , e_m$ in $V$ are:

1. Independent (mod $W$) if their images $\overline{e_1}, \cdots , \overline{e_m}$ in $V/W$ are linearly indepen-
dent. Since $\sum_i c_i e_i = 0$ in $V/W$ if and only if $\sum_i c_i e_i \in W$ in $V$, that means:

$$\sum_{i=1}^{m} c_i e_i \in W \Rightarrow c_1 = \cdots = c_m = 0 \quad (c_i \in \mathbb{F})$$

2. **Span** $V \pmod{W}$ if $\mathbb{F}\text{-span}\{e_i\} = V/W$, which means: given $v \in V$, there are $c_i \in \mathbb{F}$ such that $v - \sum_i c_i e_i \in W$, or $\overline{v} = \sum_{i=0}^{m} c_i e_i$ in $V/W$.

3. **A basis for** $V \pmod{W}$ if the images $\{e_i\}$ are a basis in $V/W$ (if and only if 1. and 2. hold).

Now, let $W \subseteq \mathbb{R}^5$ be the solution set of system

$$\begin{cases} x_1 + x_3 &= 0 \\ x_1 - x_4 &= 0 \end{cases}$$

and let $\{e_i\}$ be the standard basis in $V = \mathbb{R}^5$.

1. Find vectors $v_1, v_2$ that are a basis for $V \pmod{W}$.
2. Is $\mathcal{X} = \{e_1, e_2, e_3, v_1, v_2\}$ a basis for $V$ where $v_1, v_2$ are the vectors in (1.)?
3. Find a basis $\{f_1, f_2, f_3\}$ for the subspace $W$.

**Exercise 5**

Let

$$A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$$

and $\mathbb{F} = \mathbb{R}$.

1. If possible find a basis for $\mathbb{F}^n$ consisting of eigenvectors of $A$.
2. If successful in finding a basis, determine an invertible matrix $Q$ and a diagonal matrix $D$ such that $Q^{-1}AQ = D$.

**Exercise 6:**

For each of the following linear operator $T$, test $T$ for diagonalizability. If $T$ is diagonalizable, find a basis $\beta$ such that $[T]_\beta$ is a diagonal matrix.

1. $T : P_3(\mathbb{R}) \rightarrow P_3(\mathbb{R})$ defined by $T(f) = f' + f''$ where $f'$ and $f''$ are the first and second derivatives of $f$, respectively.
2. $T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R})$ defined by $T(ax^2 + bx + c) = cx^2 + bx + a$. 