Problem Set # 12

Justify all your answers completely (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can’t make sense of it in finite time you could lose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will lose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

Exercise 1:
If $B$ is non degenerate, symmetric over $\mathbb{F} = \mathbb{C}$. Prove that there is a bases $\mathcal{X}$ such that $[B]_\mathcal{X} = I_{n \times n}$. In coordinates, for this basis $B(x,y) = \sum_{j=1}^{n} x_j y_j$.

Exercise 2: Recall that an elementary matrix is a square matrix obtained by performing one column/row operation (switch, scale or replacement) to the identity matrix.

1. Let $E$ be an elementary $n \times n$ matrix corresponding to a column operation and let $A$ be a square $n \times n$ matrix. Describe the transformation from $A$ to $AE$ and the transformation from $A$ to $E^T A$ as an operation on the columns or the rows of $A$ compare it with the one producing $E$.

2. Prove that $A$ is invertible if and only if $A$ is a product of elementary matrices.

3. Let

$$A = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

we want to find $Q$ invertible such that $Q^T AQ = D$ where $D$ is diagonal. Use elementary operations successively and the previous question to build $Q$ as a product of elementary matrices and get an algorithm to get $D$. (First, find an elementary matrix $E_1$, such that $E_1^T AE_1$ is

$$\begin{pmatrix} 3 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & -1 \end{pmatrix}$$

What is the signature of $A$?

Exercise 3: If $\mathcal{X}, \mathcal{Y}$ are bases in $V$ and we define $G_{B,\mathcal{X}}, G_{B,\mathcal{Y}}$ as defined in class. Prove that

$$G_{B,\mathcal{Y}} = S^{-1} G_{B,\mathcal{X}} S$$
where $S = [id]_{X,Y}$

**Exercise 4:** Suppose $B : V \times V \to \mathbb{F}$ is a non degenerate bilinear form on a finite dimensional vector space. For $x \in V$ define the linear functional $l_x$ such that

$$<l_x, v> = B(v, x) \text{ for all } v \in V$$

Verify that $l_x : V \to \mathbb{F}$ is in fact a linear map, and the correspondence $\Phi : V \to V^*$ given by $\Phi(x) = l_x$ is an $\mathbb{F}$-linear isomorphism between $V$ and the dual space $V^*$.

**Exercise 5:** Let $B : V \times V \to \mathbb{F}$ be a nondegenerate symmetric bilinear form on a finite dimensional vector space. If $W$ is a subspace in $V$ define $W^\perp = \{v \in V : B(v, w) = 0 \text{ for all } w \in W\}$. Prove that

1. $\dim(W) + \dim(W^T) = \dim(V)$
2. $W_1 \subseteq W_2 \Rightarrow (W_2)^\perp \subseteq (W_1)^\perp$
3. $(W^\perp)^\perp = W$.

**Exercise 6:**
If $T : V \to W$ is a linear map between finite dimensional inner product spaces the adjoint $T^* : W \to V$ is the unique $\mathbb{F}$-linear operator such that

$$(T^*(w), v)_V = (w, T(v))_W, \forall v \in V, w \in W$$

Prove that

1. $\text{Range}(T^*) = \text{Ker}(T)^\perp$
2. $\text{Range}(T) = \text{Ker}(T^*)^\perp$
3. If $T : V \to W$ and $S : W \to U$ are linear maps between finite dimensional inner product spaces, show that $(S \circ T)^* : U \to V$ is equal to $T^* \circ S^*$.
4. Show that $T^{**} = T$.

**Exercise 7:**
Prove that every positive definite operator $P$ on an inner product space has a unique positive definite square root $A = \sqrt{P}$ i.e. $A \geq 0$ and $A^2 = T$.

**Exercise 8:**
For the self-adjoint matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$
1. Is $A$ orthogonally diagonalizable on $\mathbb{C}^3$.

2. Determine the projections $p_\lambda (\lambda \in Sp(A))$ as operators on $\mathbb{C}^3$.

**Exercise 9:**
Let $H : V \times V \to \mathbb{F}$ with $\dim(V) > 1$, bilinear form. Is it true or false that for any $x \in V$, there is $y \in V$ such that $y \neq 0$ but $H(x, y) = 0$?

**Exercise 10:**
Show that $H : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is an antisymmetric bilinear form, where

$$H(x, y) = \det\left(\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix}\right)$$

with $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$;