Quizz #1

Friday 4 September during recitation

Problems:
Consider \( p(x) = x^3 + x - 2 \) a polynomial of \( \mathbb{R}[x] \)

1. Find the irreducible components of \( p(x) \) in \( \mathbb{Q}[x] \).
2. Find the irreducible components of \( p(x) \) in \( \mathbb{Z}/2\mathbb{Z}[x] \).

Write \( R = \mathbb{Q}[x]/(p(x)) \) for the quotient of \( \mathbb{Q}[x] \) by the ideal generated by \( p(x) \)

3. Prove that a representative of some element of the quotient \( R \) can be expressed as a polynomial of degree at most 2. What represents the image of \( (x^3 + 17x^2 + 20x + 3)p(x) + 1 \) in \( R \).
4. Describe the ideals of \( R \).
5. Is \( R \) a field?

(Remark: You can answer to Questions 1., 2., 3. without any other question, maybe just questions 4. and 5. are not independents from the other.)

Solution:

1. We have that \( p(x) = (x^2 - 2x + 1) + (x^3 - x^2 + 3x - 3) = (x - 1)^2 + (x^2(x - 1) + 3(x - 1)) = (x - 1)^2 - (x^2 + 3)(x - 1) = (x - 1)(x - 1 + x^2 + 3) = (x - 1)(x^2 + x + 2). \)
   \( (x - 1) \) is a linear polynomial thus irreducible. \( x^2 + x + 2 \) has a discriminant equal to \( 1^2 - 4 \times 2 = -7 \) so \( x^2 + x + 2 \) are no reals roots then a fortiori no rationals roots. Thus it is irreducible.
2. we have that \( 2 = 0 \) in \( \mathbb{Z}/2\mathbb{Z} \), so \( p(x) = x(x-1)(x+1) \) thus \( x, (x-1) \) and \( (x+1) \) are the irreducible component of \( p(x) \).
3. Let \( [q(x)] \in R \) where \( q(x) \in \mathbb{Q}[x] \), if \( q(x) \) is a polynomial of degree at most 2, then there is nothing to prove. If not, then we know that we can make the euclidean division of \( q(x) \) by \( p(x) \), there is polynomials \( \alpha(x) \) and \( \beta(x) \) such that \( q(x) = \alpha(x)p(x) + \beta(x) \) with \( \deg(\beta(x)) < \deg(p(x)) = 3 \). But by definition of a quotient, \( [q(x)] = [\beta(x)] \) and we can choose \( \beta(x) \) as representative of the initial class, it has degree at most 2. \( (x^3 + 17x^2 + 20x + 3)p(x) + 1 \) correspond to the unit \([1]\) in \( R \).
4. We know that the ideals of $\mathbb{Q}[x]$ are principals, since $\mathbb{Q}[x]$ is a principal ideal domain. Then the ideals containing $p(x)$ are the one generated by $(x - 1), (x^2 + x + 2), (x - 1)(x^2 + x + 2)$ and $\mathbb{Q}[x]$ are in one-to-one correspondence to the one of $R$, which are then exactly $\mathbb{Q}[x]/(p(x)), (x - 1)/(p(x)), (x^2 + x + 2)/(p(x))$ and \{0\}.

5. $R$ is not a field since $p(x)$ is not irreducible or since there is more than the two ideals \{0\} and $\mathbb{Q}[x]/(p(x))$.

\footnote{\textit{\textdagger} = easy, \textit{\textdaggerdbl} = medium, \textit{\textdaggerdash} = challenge}