Problem Set #3

Due Monday 30 September in Class

Exercise 1 (⋆⋆): 4 points
Let $p$ be a prime number and define the cyclotomic polynomial $\Phi_p$ of order $p$ by

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \ldots + x + 1 \in \mathbb{Z}[x]$$

Show that $\Phi_p(x)$ is irreducible over $\mathbb{Z}$. (Hint: First compute $\Phi_p(x + 1)$ using the binomial formula and prove it is irreducible using a criterium of the course, then conclude about the irreducibility of $\Phi_p(x)$.)

Solution:
Note first that

$$\Phi_p(x + 1) = \frac{(x + 1)^p - 1}{x} = \sum_{i=1}^{p} \binom{p}{i} x^{i-1}$$

We have that $p|\binom{p}{i}$ for all $i \in \{1, 2, \ldots, p-1\}$ and $p^2 \nmid \binom{p}{1} = p$. Therefore by Eisenstein’s Criterion, we have that $\Phi_p(x + 1)$ is irreducible over $\mathbb{Q}$ and hence over $\mathbb{Z}$.
Lastly, note that if $\Phi_p(x)$ were reducible, then $\Phi_p(x)$ is also irreducible over $\mathbb{Z}$.

Exercise 2 (⋆): 4 points
Let $I = (2, x)$ be the ideal of $\mathbb{Z}[x]$ generated by 2 and $x$. Show that $I$ is not a principal ideal. (Remark: This proves that $\mathbb{Z}[x]$ is not a principal ideal ring, so in particular it is not Euclidean.)

Solution:
We not that $2 \not\subset (x)$ and that $(x) \not\subset (2)$. Suppose that there exists $\alpha \in \mathbb{Z}[x]$ for $(2, x) = (\alpha)$. Then we have $2 \in (\alpha)$ and $x \in (\alpha)$ from which we deduce that $\alpha | 2$ and $\alpha | x$. But if $\alpha | 2$, then $\alpha \in \{\pm 1, \pm 2\}$. However, we observe that $\pm 1 \not\in (2, x)$ and $\pm 2 \nmid x$, we get to a contradiction.

Exercise 3 (⋆⋆): 4 points

1. Prove that $p(x) = x^4 + 1$ is irreducible over $\mathbb{Q}$ using Eisenstein criterion on $p(x+1)$.
2. Find the irreducible factor of $x^8 - 1$ in $\mathbb{Q}[x]$. 
**Solution:**

1. Observe that $p(x+1) = (x+1)^4 + 1 = x^4 + 4x^3 + 6x^2 + 4x + 2$ verifies the Eisenstein criterion for $p = 2$.

2. $x^8 - 1 = (x^4 - 1)(x^4 + 1) = (x^2 - 1)(x^2 + 1)(x^4 + 1) = (x-1)(x+1)(x^2+1)(x^4+1)$. This is a factorization of $x^8 - 1$ into irreducible in $\mathbb{Q}[x]$ The first two are linear then irreducible. The third factor has roots $\pm i \notin \mathbb{Q}$; hence it is irreducible. The last one was studied in point 1.

**Exercise 4 (⋆): 4 points**
Determine if the following sets are subspaces of $\mathbb{R}^3$ (Give a complete justification to your answer):

1. $V = \{(x,y,z) | x,y,z \in \mathbb{R} \text{ and } x + y = 1\}$.
2. $V = \{(x,y,z) | x,y,z \in \mathbb{R} \text{ and } x + 2y + z = 0\}$.

**Solution:**

1. $(0,0,0)$ is not in $V$, $V$ is not a subspace of $\mathbb{R}^3$.

2. For each $r,s \in \mathbb{R}$, and for each $(x,y,z)$ and $(x',y',z')$ in $V$, we have $x + 2y + z = 0$ and $x' + 2y' + z' = 0$. We want to prove that $r(x,y,z) + s(x',y',z') = (rx + sx', ry + sy', rz + sz')$ is in $V$. But, $rx + sx' + 2(ry + sy') + (rz + sz') = r(x + 2y + z) + s(x' + 2y' + z') = 0$ by the assumptions.

**Exercise 5 (⋆): 4 points**
Let $F \subset K$, both fields, and consider $K$ as a vector space over $F$. Let $\alpha \in K - \{0\}$, Prove that the map $T_\alpha : K \to K$ given by $T_\alpha(\beta) = \alpha \beta$ is a homomorphism. Prove further that it is an isomorphism between $K$ and itself.

**Solution:**

We have

$T_\alpha(\beta_1 + \beta_2) = \alpha(\beta_1 + \beta_2) = \alpha \beta_1 + \alpha \beta_2 = T_\alpha(\beta_1) + T_\alpha(\beta_2)$

and for any $\kappa \in F, \beta \in K$

$T_\alpha(\kappa \beta) = \alpha \kappa \beta = \kappa T_\alpha(\beta)$

It is an isomorphism because if we consider the map $U_\alpha$ defined by $U_\alpha(\beta) = \alpha^{-1} \beta$, it is its inverse. Indeed, it is easy to check that $U_\alpha T_\alpha = T_\alpha U_\alpha = Id_K$.  

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1(⋆) = easy, (⋆⋆) = medium, (⋆⋆⋆) = challenge