Problem 1: (**) 60 points

Let
\[ \mathbb{Q}(\sqrt{2}) := \{ a + b\sqrt{2} | a, b \in \mathbb{Q} \} \]

We admit that \( \mathbb{Q}(\sqrt{2}) \) is a subring of \( \mathbb{R} \).

1. Show that every non-zero element of \( \mathbb{Q}(\sqrt{2}) \) is invertible in \( \mathbb{Q}(\sqrt{2}) \). (Remark: Then, we have proven that \( \mathbb{Q}(\sqrt{2}) \) is a field.)

2. Let \( a \) be a rational number and \( b \) a positive rational such that \( \sqrt{b} \) is irrational. Prove that if \( a + \sqrt{b} \) is a root of some polynomial with rational coefficients then \( a - \sqrt{b} \) is also a root of this polynomial. (Hint: Compute the Euclidean division by \( ((x - a)^2 - b) \) on \( \mathbb{Q}[x] \) and prove that \( ((x - a)^2 - b) \) divides \( p(x) \).)

3. Show that there is an isomorphism
\[ \mathbb{Q}[x]/(x^2 - 2) \simeq \mathbb{Q}(\sqrt{2}) \]
Problem 2: (∗) 15 points If $r \in R$, $R$ is an integral domain and $r^2 = 1$, prove $r = \pm 1$.

Problem 3: (∗∗) 30 points (The questions are independent.)

1. Find the remainder and the quotient upon division of $x^3$ by $x^2 + x + 4$ in $(\mathbb{Z}/2\mathbb{Z})[x]$.

2. List all the irreducible monic quadratic polynomials in $(\mathbb{Z}/2\mathbb{Z})[x]$. Justify your answer.
Problem 4: (⋆) 20 points
Prove that $\sqrt[5]{\frac{17}{25}}$ is not a rational number. (Hint: It may be helpful to consider the polynomial $f(x) = 25x^5 - 17$.)

Problem 5: (⋆) 30 points Define $W = \{f \in \mathbb{R}_3[x] | f'(0) = 0\}$.

1. Prove that $W$ is a subspace of $\mathbb{R}_3[x]$ - the space of the polynomials of degree less or equal to 3;

2. Find a basis for $W$ (and prove that it is a basis).
Problem 6: (**) 20 points Prove that the two subrings of \( \mathbb{Z} \): \( 2\mathbb{Z} \) and \( 3\mathbb{Z} \) are not isomorphic as rings.

Problem 7: (*) 35 points

1. Let \( n \in \mathbb{Z}_{\geq 2} \) and suppose that \( a_nx^n + \ldots + a_1x + a_0 \in \mathbb{Z}[x] \) has factor \( ax + b \). Show that \( a|a_n \) and \( b|a_0 \).

2. **Bonus**: Suppose \( \alpha \) is a rational root of a monic polynomial in \( \mathbb{Z}[x] \). prove that \( \alpha \) is an integer. (Hint: write \( \alpha = p/q \) where \( p \) and \( q \) are coprime, prove that then \( q|p \) and conclude.)