Problem Set # 9

Due Friday November 22th in recitation

Exercise 1(⋆): 30 points
Let $K/L$ be a field extension and $\alpha \in K$. Prove that if $[F(\alpha) : F]$ is odd then $F(\alpha) = F(\alpha^2)$.

Exercise 2(⋆): 60 points
A field of prime characteristic $p$ is perfect if the map $F \rightarrow F$ given by $\alpha \mapsto \alpha^p$ is surjective.

1. Show every finite field is perfect.
2. Let $F$ be an arbitrary field of characteristic $p \neq 0$. Show that the field of rational function

$$F(x) = \left\{ \frac{f(x)}{g(x)} : \forall f \in K[x], g \in K[x]^\times \right\}$$

where $K[x]^\times = K[x]\{\text{the null polynomial}\}$ is not perfect. (Hint: Consider the polynomial $x \in F(x)$.)

Exercise 3(⋆): 60 points
1. Suppose that $[L : K] = p$ is a prime number. Prove that $L/K$ is a simple extension i.e. there is $\alpha \in L$ such that $L = K(\alpha)$ (Hint: You can choose any $\alpha \in L \setminus F$).
2. Let $L/K$ be a finite extension, and let $p(x)$ be an irreducible polynomial in $K[x]$ with $\text{deg}(p(x)) \geq 2$. Prove by contradiction that, if $\text{deg}(p)$ and $[L : K]$ are coprime, then $p(x)$ has no zeros in $L$. (Hint: If $\alpha \in L$ is a root of $p(x)$, then consider the field $K(\alpha)$.)

Exercise 4(⋆): 30 points
Let $K$ be a finite extension field of $F$, then any endomorphism of $K$ over $F$ is an automorphism.

Exercise 5(⋆): 20 points Prove that the ring $F_2[x]/(x^3 + x + 1)$ is a field, but that $F_3[x]/(x^3 + x + 1)$ is not a field.