Problem Set # 8

Due Friday November 8th in Recitation

Exercise 1(⋆): 40 points
Prove that if \( x \) and \( y \) are two constructible numbers, then \( x \pm y, xy, x/y \) with \( y \neq 0 \) are also constructible.

Exercise 2(⋆): 80 points
Let \( F \) be any subfield of the real number

1. Prove that a line in \( F \) has an equation of the form \( y = mx + p \) with \( m \) and \( p \) in \( F \).

2. Prove that a circle in \( F \) has an equation of the form \((x - a)^2 + (x - b)^2 = c\) with \( a, b \) and \( c \) in \( F \).

3. Prove that two lines in \( F \), which intersect in the real plane, intersect at a point in the plane of \( F \).

4. Prove that a line in \( F \) and a circle in \( F \) which intersect in the real plane do so at a point either in the plane of \( F \) or in the plane of \( F(\sqrt{\gamma}) \) where \( \gamma \) is a positive number in \( F \).

Exercise 3(⋆): 20 points
Prove that square roots of positive rational number are constructible.

Exercise 4(⋆): 20 points
Prove that the polynomial \( f(x) = 8x^3 - 6x - 1 \) is irreducible over the field of rational numbers. (Hint: Argue by contradiction, let \( r \) be a root of \( f(x) \) in \( \mathbb{Q} \) then prove that \( v = 2r - 1 \) is a root of the polynomial \( X^3 + 3X^2 - 3 \) and conclude.)

Exercise 5(⋆⋆): 40 points

1. Prove that \( f(x) = x^3 + x^2 - 2x - 1 \) is irreducible in \( \mathbb{Q} \) by contradiction.

2. Prove that \( 2\cos(2\pi/7) \) satisfies \( f(x) = x^3 + x^2 - 2x - 1 \). (Hint: \( 2\cos(2\pi/7) = e^{2i\pi/7} + e^{-2i\pi/7} \).)