Problem Set #3

Due Monday 23 September in Class

**Exercise 1 (**⋆⋆**)** 4 points:
Let \( p \) be a prime number and define the **cyclotomic polynomial** \( \Phi_p \) of order \( p \) by
\[
\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + ... + x + 1 \in \mathbb{Z}[x]
\]
Show that \( \Phi_p(x) \) is irreducible over \( \mathbb{Z} \). (Hint: First compute \( \Phi_p(x+1) \) using the binomial formula and prove it is irreducible using use Eisenstein’s Criterion, then conclude about the irreducibility of \( \Phi_p(x) \).)

**Exercise 2 (**⋆**)** 4 points:
Let \( I = (2, x) \) be the ideal of \( \mathbb{Z}[x] \) generated by 2 and \( x \). Show that \( I \) is not a principal ideal. (Remark: This proves that \( \mathbb{Z}[x] \) is not a principal ideal ring, so in particular it is not Euclidean.)

**Exercise 3 (**⋆**)** 4 points:
1. Prove that \( p(x) = x^4 + 1 \) is irreducible over \( \mathbb{Q} \) using Eisenstein criterion on \( p(x+1) \).
2. Find the irreducible factors of \( x^8 - 1 \) in \( \mathbb{Q}[x] \).

**Exercise 4 (**⋆**)** 4 points:
Determine if the following sets are subspaces of \( \mathbb{R}^3 \) (Give a complete justification to your answer):
1. \( V = \{(x, y, z)|x, y, z \in \mathbb{R} \text{ and } x + y = 1\} \).
2. \( V = \{(x, y, z)|x, y, z \in \mathbb{R} \text{ and } x + 2y + z = 0\} \).

**Exercise 5 (**⋆**)** 4 points:
Let \( F \subset K \), both fields, and consider \( K \) as a vector space over \( F \). Let \( \alpha \in K - \{0\} \). Prove that the map \( T_\alpha : K \to K \) given by \( T_\alpha(\beta) = \alpha \beta \) is a homomorphism. Prove further that it is an isomorphism between \( K \) and itself. \(^1\)

\(^1(⋆) = \text{easy}, (⋆⋆) = \text{medium}, (⋆⋆⋆) = \text{challenge} \)