Problem Set # 12

Due Wednesday december 11th in class

Exercise 1(⋆): 40 points
Let $\alpha, \beta, \gamma$ denote the roots of the polynomial $x^3 + x + 1$.
1. Find $\alpha + \beta + \gamma, \alpha \beta + \alpha \gamma + \beta \gamma, \alpha \beta \gamma$.
2. Find $\alpha^2 + \beta^2 + \gamma^2$.
3. Find $\alpha^2 \beta + \alpha^2 \gamma + \beta^2 \alpha + \beta^2 \gamma + \gamma^2 \alpha + \gamma^2 \beta$.
4. Find $\alpha^3 + \beta^3 + \gamma^3$.

Exercise 2(⋆): 40 points
Let $K/\mathbb{Q}$ be a normal extension and assume the Galois Group $Gal(K, \mathbb{Q})$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z}, +)$ where $n = 2^t$. Prove that there exists a tower $\mathbb{Q} = F_0 \subset F_1 \subset \cdots F_t = K$ where $F_{i+1} = F_i(\sqrt[n]{\alpha_i})$ for some $\alpha_i \in F_i$.

Exercise 3(⋆): 40 points
Let $K/\mathbb{Q}$ be a normal extension, set $G = Gal(K, \mathbb{Q})$. Let $H$ be a subgroup of $G$. Let $a = |H|, s = |G|$. Let $\alpha \in K$. Set $\gamma = \sum_{\sigma \in H} \sigma(\alpha)$.
1. Show that $\tau(\gamma) = \gamma$ for all $\tau \in H$.
2. Show that $H \subseteq G_{\mathbb{Q}(\gamma)}$.
3. Deduce an upper bound on $[\mathbb{Q}(\gamma) : \mathbb{Q}]$.

Exercise 4(⋆): 40 points
Let $\alpha$ be a root of $f(x) = x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x]$.
1. Show $f(x)$ is irreducible over $\mathbb{Q}$. Note: You should assume this in what follows.
2. Find $[\mathbb{Q}(\alpha) : \mathbb{Q}]$.
3. Set $\beta = -1/(\alpha + 1)$. Find $\beta$ in the form $a + b\alpha + c\alpha^2$.
4. Show that $f(\beta) = 0$. (Bit of grunt work here!)
5. Find $\gamma \in \mathbb{Q}(\alpha), \gamma, \beta, \alpha$ distinct, with $f(\gamma) = 0$. (Idea: If $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$ then $\alpha + \beta + \gamma$ is determined.)
6. Deduce that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is normal.

7. List all of the $\mathbb{Q}$-automorphisms of $\mathbb{Q}(\alpha)$. What familiar group is $Gal(\mathbb{Q}(\alpha), \mathbb{Q})$ isomorphic to?.

8. Let $K_0$ be the fixed field of $Gal(\mathbb{Q}(\alpha), \mathbb{Q})$. Find $[\mathbb{Q}(\alpha) : K_0]$. What is $K_0$?

Exercise 5(⋆): 40 points

Let $f(x) = x^4 + 1$, which you may assume is irreducible over $\mathbb{Q}$. Set $\beta = e^{\pi i / 4}$.

1. Show $f(\beta) = 0$.

2. Draw a nice picture on the Complex Plane with the four roots of $f(x)$ clearly marked.

3. Find the four roots of $f(x)$ in terms of $\beta$. Call them $\beta, \gamma, \lambda, \kappa$.

4. Set $L = \mathbb{Q}(\beta)$. Show that $L$ is the splitting field of $f(x)$ over $\mathbb{Q}$.

5. Describe all of the automorphisms $\sigma \in Gal(L, \mathbb{Q})$. For each one, give how it permutes $\beta, \gamma, \lambda, \kappa$.

6. Which one of the above $\sigma$ is complex conjugation?

7. Give the group table for $Gal(L, \mathbb{Q})$. What familiar group is $Gal(L, \mathbb{Q})$ isomorphic to?

8. For each $\sigma \in Gal(L, \mathbb{Q})$ describe the set of $\eta \in L$ with $\sigma(\eta) = \eta$. (Write $\eta = a + b\beta + c\beta^2 + d\beta^3$.)