STATISTICAL PROPERTIES OF DYNAMICAL SYSTEMS WITH SOME HYPERBOLICITY

LAI-SANG YOUNG*

Department of Mathematics
University of California, Los Angeles
Los Angeles, CA 90095
Email: lsy@math.ucla.edu

This paper is about the ergodic theory of attractors and conservative dynamical systems with hyperbolic properties on large parts (though not necessarily all) of their phase spaces. The main results are for discrete time systems. To put this work into context, recall that for Axiom A attractors the picture has been fairly complete since the 1970's (see [S1], [B], [R2]). Since then much progress has been made on two fronts: there is a general nonuniform theory that deals with properties common to all diffeomorphisms with nonzero Lyapunov exponents ([O], [P1], [Ka], [LY]), and there are detailed analyses of specific kinds of dynamical systems including, for example, billiards, 1-dimensional and Hénon-type maps ([S2], [BSC]; [HK], [J]; [BC2], [BY1]).

Statistical properties such as exponential decay of correlations are not enjoyed by all diffeomorphisms with nonzero Lyapunov exponents. The goal of this paper is a systematic understanding of these and other properties for a class of dynamical systems larger than Axiom A. This class will not be defined explicitly, but it includes some of the much studied examples. By looking at regular returns to sets with good hyperbolic properties, one could give systems in this class a simple dynamical representation. Conditions for the existence of natural invariant measures, exponential mixing and central limit theorems are given in terms of the return times. These conditions can be checked in concrete situations, giving a unified way of proving a number of results, some new and some old. Among the new results are the exponential decay of correlations for a class of scattering billiards and for a positive measure set of Hénon-type maps.

The dynamical picture we wish to focus on is the following. Let f be the map in question, and suppose that f admits a "horseshoe" Λ with infinitely many branches and variable return times. More precisely, Λ has a product structure in the sense

*This research is partially supported by NSF

that it is the intersection of two transversal families of stable and unstable manifolds. Dynamically, it is the disjoint union of a countable number of sets Λ_i with the property that each Λ_i extends fully in the stable direction, and for each i there is an integer R_i such that f^{R_i} maps Λ_i onto a subset of Λ, crossing it completely in the unstable direction. Let R be the return time function, i.e. $R \mid \Lambda_i = R_i$. We prove the following:

(1) if Λ intersects its unstable manifolds in positive Lebesgue measure sets, and $\int R d\mu^u < \infty$ where μ^u denotes Lebesgue measure on unstable manifolds, then f admits a Sinai-Ruelle-Bowen measure ν; and
(2) if additionally $\mu^u\{R > n\}$ decreases exponentially with n, then (f, ν) has exponential decay of correlations for Hölder continuous test functions provided that the usual aperiodicity conditions are met; under the same conditions the central limit theorem also holds.

Conceptually, (2) says that under the usual aperiodicity assumptions, a sufficient condition for exponential mixing is that "most" arbitrarily small pieces of unstable manifolds grow to *a fixed size* at exponential speeds. Precise formulations of these results are given in Section 1. We remark that our setup bears a certain resemblance to countable state Markov chains for which the corresponding results are also valid. We must emphasize, however, that these results are for discrete time systems; (2) above is false for flows; see e.g. [R3].

In order to apply these "abstract" results to specific dynamical systems, we must ask the following questions: given f with some hyperbolicity, does Λ with $\mu^u(\Lambda) > 0$ exist, how to find it, and how to determine the nature of R? We do not know how to deal with general diffeomorphisms, so let us specialize to the following situation: suppose there is a recognizable set away from which f is uniformly hyperbolic, and suppose that when an orbit passes near this set it suffers a certain setback in its hyperbolicity from which it will attempt to recover. Assume further that we have quantitative knowledge of both the setback and recovery. The methods of this paper will suggest that under these conditions

(a) there is a systematic way of choosing Λ, namely by fixing a box, taking points in it that approach the "bad set" not faster than a certain rate , and running the system until the various parts of Λ return as desired;
(b) the speeds with which orbits recover from the influence of the "bad set" are reflected in $\mu^u(\Lambda)$ and in the nature of the return time function R.

To give some examples of "bad sets", for billiards they might be thought of as directions that give rise to trajectories making tangential contacts with the boundary of the table, whereas for Hénon-type maps it is clearly the "turns" that spoil hyperbolicity. *Our scheme of proof is potentially applicable to dynamical systems for which the mechanisms that cause hyperbolicity to fail are known and the source of nonhyperbolicity is localized.*

In Part I of this paper we will prove (1) and (2) assuming the existence of a "horseshoe" with infinitely many branches and variable return times. In Part II we will illustrate (a) and (b) for several relatively simple situations. In each case it will be shown that the return time function has the desired exponential estimate. It then follows immediately from the results of Part I that they admit SRB measures, have exponential decay of correlations etc.

TABLE OF CONTENTS

The work discussed in Section 10 is joint with M. Benedicks; the proofs will appear elsewhere.

Acknowledgements. Some of the ideas of this paper were conceived while I was working on the Hénon maps and I would like to thank Benedicks for his input. My thanks go also to N. Chernov for sharing with me his expertise on billiards and for his comments on the manuscript. Finally, I wish to acknowledge the hospitality of MSRI, where much of this work was done.

PART I. AN ABSTRACT MODEL AND ITS MIXING PROPERTIES

1. Setting and Assertions

Let $f : M \circlearrowleft$ be a $C^{1+\varepsilon}$ diffeomorphism of a finite dimensional Riemannian manifold M. In applications we will allow f to have discontinuities or singularities, but these "bad" parts will not appear in the picture we are about to describe. Thus as far as Part I is concerned we may assume that f and f^{-1} are defined on all of M. Let $d(\cdot, \cdot)$ denote the distance between points. Riemannian measure on M will be denoted by μ; and if $W \subset M$ is a submanifold, then μ_W denotes the measure on W induced by the restriction of the Riemannian structure to W. The basic object of interest here consists of a set $\Lambda \subset M$ with a "hyperbolic product structure" and a return map f^R from Λ to itself. Precise definitions are given in 1.1 and 1.2; the required properties are listed in (P1)-(P5); and the main results of Part I are stated in 1.4.

1.1. A "horseshoe" with infinitely many branches and variable return times.

We begin with some formal definitions.

An embedded disk $\gamma \subset M$ is called an *unstable manifold* or *unstable disk* if $\forall x, y \in \gamma,\ d(f^{-n}x,\ f^{-n}y) \to 0$ exponentially fast as $n \to \infty$; it is called a *stable manifold* or *stable disk* if $\forall x, y \in \gamma,\ d(f^n x,\ f^n y) \to 0$ exponentially fast as $n \to \infty$.

We say that $\Gamma^u = \{\gamma^u\}$ is a *continuous family of C^1 unstable disks* if the following hold:

* K^s is an arbitrary compact set; D^u is the unit disk of some $\mathbb{R}^n$;
* $\Phi^u : K^s \times D^u \to M$ is a map with the property that
  - Φ^u maps $K^s \times D^u$ homeomorphically onto its image,
  - $x \mapsto \Phi^u \mid (\{x\} \times D^u)$ is a continuous map from K^s into $\mathrm{Emb}^1(D^u, M)$, the space of C^1 embeddings of D^u into M,
  - γ^u, the image of each $\{x\} \times D^u$, is an unstable disk.

Continuous families of C^1 stable disks are defined similarly.

Definition 1. We say that $\Lambda \subset M$ has a *hyperbolic product structure* if there exist a continuous family of unstable disks $\Gamma^u = \{\gamma^u\}$ and a continuous family of stable disks $\Gamma^s = \{\gamma^s\}$ such that

(i) $\dim \gamma^u + \dim \gamma^s = \dim M$;
(ii) the γ^u-disks are transversal to the γ^s-disks with the angles between them bounded away from 0;
(iii) each γ^u-disk meets each γ^s-disk in exactly one point; and
(iv) $\Lambda = (\cup\gamma^u) \cap (\cup\gamma^s)$.

We will assume throughout Part I that

(P1) *there exists $\Lambda \subset M$ with a hyperbolic product structure and with $\mu_\gamma\{\gamma \cap \Lambda\} > 0$ for every $\gamma \in \Gamma^u$.*

Next we define a return map on Λ that gives it the structure of a "horseshoe" – except that unlike the standard horseshoe this one has infinitely many branches returning at variable times. Let Γ^u and Γ^s be the defining families for Λ. A subset $\Lambda_0 \subset \Lambda$ is called an *s-subset* if Λ_0 also has a hyperbolic structure and its defining families can be chosen to be Γ^u and Γ^s_0 with $\Gamma^s_0 \subset \Gamma^s$; *u-subsets* are defined analogously. For $x \in \Lambda$, let $\gamma^u(x)$ denote the element of Γ^u containing x. We assume:

(P2) *there are pairwise disjoint s-subsets $\Lambda_1, \Lambda_2, \ldots \subset \Lambda$ with the properties that*

- *on each γ^u-disk, $\mu_{\gamma^u}\{(\Lambda - \cup\Lambda_i) \cap \gamma^u\} = 0$;*
- *for each i, $\exists R_i \in \mathbb{Z}^+$ s.t. $f^{R_i}\Lambda_i$ is a u-subset of Λ; we require in fact that for all $x \in \Lambda_i$, $f^{R_i}(\gamma^s(x)) \subset \gamma^s(f^{R_i}x)$ and $f^{R_i}(\gamma^u(x)) \supset \gamma^u(f^{R_i}x)$;*

- *for each n, there are at most finitely many i's with $R_i = n$;*
- $\min R_i \geq$ *some* $R_0 > 1$ *depending only on* f. (R_0 depends in fact only on the constants C and α in (P3)-(P5).)

1.2. Separation times and derivative estimates.

For every pair $x, y \in \Lambda$, we assume there is a notion of *separation time* denoted by $s_0(x,y)$. The nature of the separation depends on the application in question and will not be specified in this abstract part. Some examples of separation are when two points move a certain distance apart, or land on opposite sides of a discontinuity curve, or that their derivatives cease to be comparable. We say $s_0(x,y) = n$ if the orbits of x and y are "together" through their n^{th} iterates and $f^{n+1}x$ and $f^{n+1}y$ are "separated". We assume that

(i) $s_0(\cdot,\cdot) \geq 0$ and depends only on the γ^s-disks containing the two points;
(ii) the maximum number of orbits starting from Λ that are pairwise separated before time n is finite for each n;
(iii) for $x, y \in \Lambda_i$, $s_0(x,y) \geq R_i + s_0(f^{R_i}x,\ f^{R_i}y)$; in particular, $s_0(x,y) \geq R_i$;
(iv) for $x \in \Lambda_i, y \in \Lambda_j, i \neq j$ but $R_i = R_j$, we have $s_0(x,y) < R_i - 1$.

Conditions (iii) and (iv) describe the relation between $s_0(\cdot,\cdot)$ and returns to Λ, namely that points in the same Λ_i must not separate before they return, while points in distinct Λ_i's must first separate if they are to return simultaneously. We remark also that in the proofs to follow, it is only necessary that (ii) holds for $n \leq R_0$. (See the remark at the end of 3.5.)

We now state the required analytic estimates that accompany the topological picture in 1.1. Let f^u denote the restriction of f to γ^u-disks, and let $\det(Df^u)$ be the Jacobian of $D(f^u)$.

We assume there exist $C > 0$ and $\alpha < 1$ s.t. the following hold for all $x, y \in \Lambda$:

(P3) Contraction along γ^s-disks. *For $y \in \gamma^s(x)$,* $d(f^n x,\ f^n y) \leq C\alpha^n \quad \forall n \geq 0$.

(P4) Backward contraction and distortion along γ^u. *For $y \in \gamma^u(x)$ and $0 \leq k \leq n < s_0(x,y)$, we have*

(a) $d(f^n x,\ f^n y) \leq C\alpha^{s_0(x,y)-n}$;

(b)

$$\log \prod_{i=k}^{n} \frac{\det Df^u(f^i x)}{\det Df^u(f^i y)} \leq C\alpha^{s_0(x,y)-n}.$$

(P5) Convergence of $D(f^i|\gamma^u)$ and absolute continuity of Γ^s.

(a) *For $y \in \gamma^s(x)$,*

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i x)}{\det Df^u(f^i y)} \leq C\alpha^n \quad \forall n \geq 0.$$

(b) *For $\gamma, \gamma' \in \Gamma^u$, if $\Theta : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ is defined by $\Theta(x) = \gamma^s(x) \cap \gamma'$, then Θ*

is absolutely continuous and

$$\frac{d(\Theta_*^{-1}\mu_{\gamma'})}{d\mu_\gamma}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i x)}{\det Df^u(f^i \Theta x)}.$$

(In practice, (P5)(b) is usually a consequence of (P3)-(P5)(a). To make this a logical implication in an abstract setting, however, requires more technical formulation of the other conditions than we would like to give.)

(P3)-(P5) are standard for Axiom A attractors. We wish to stress however that they are strictly less stringent than uniform hyperbolicity: we allow oscillatory behavior along γ^s as long as the cumulative contraction starting from Λ is uniform, and the backward contraction conditions along γ^u are imposed only at certain checkpoints allowing for a variety of behaviors in between. This is what allows us to include, for example, the Hénon maps.

1.3. A Markov extension of f.

Our next step is to construct an extension of $f : \underset{n\geq 0}{\cup} f^n\Lambda \circlearrowleft$ which has on it a natural Markov partition with a countable number of states. By an extension of $f : \cup f^n\Lambda \circlearrowleft$ we refer to a dynamical system $F : \Delta \circlearrowleft$ for which there is a projection map $\pi : \Delta \to \cup f^n\Lambda$ satisfying $f \circ \pi = \pi \circ F$. In general π will not be $1-1$.

Let $R : \Lambda \to \mathbb{Z}^+$ be the return time function, i.e. $R|\Lambda_i = R_i$, and let $f^R : \Lambda \circlearrowleft$ denote the return map with $f^R|\Lambda_i = f^{R_i}|\Lambda_i$. In ergodic theory there is a standard construction called a special flow built over a map under a function. Our extension $F : \Delta \circlearrowleft$ will be the discrete time version of the special flow built over $f^R : \Lambda \circlearrowleft$ under R. More precisely, let

$$\Delta \overset{\text{def}}{=} \{(x,\ell) : x \in \Lambda;\ \ell = 0, 1, \ldots, R(x) - 1\}$$

and define

$$F(x,\ell) = \begin{cases} (x, \ell+1) & \text{if } \ \ell + 1 < R(x) \\ (f^R x, 0) & \text{if } \ \ell + 1 = R(x). \end{cases}$$

An equivalent but less formal way of looking at Δ is to view it as the disjoint union $\overset{\infty}{\underset{\ell=0}{\cup}} \Delta_\ell$ where Δ_ℓ consists of those pairs $(x,\ell) \in \Delta$ the second coordinate of which is ℓ. We picture Δ as a tower and refer to Δ_ℓ as the ℓ^{th} level of the tower. Clearly, Δ_ℓ is a copy of $\{x \in \Lambda : R(x) > \ell\}$; we let $\iota_\ell : \{x \in \Lambda : R(x) > \ell\} \to \Lambda_\ell$ denote this canonical identification. It is clear that $\iota_\ell^{-1}(\Delta_\ell)$ is the union of a collection of Λ_i's.

We construct a Markov partition $\mathcal{D} = \{\Delta_{\ell,j}\}$ for $F : \Delta \circlearrowleft$ as follows. Let $\mathcal{D}|\Delta_0$ be the trivial partition containing a single element. Assume inductively that $\mathcal{D}|\Delta_\ell$ has been constructed and has the following properties:

(i) it is a finite partition and its elements are labeled $\Delta_{\ell,j},\ j = 1, 2, \cdots, j_\ell$;
(ii) for each j, $\iota_\ell^{-1}(\Delta_{\ell,j})$ is the union of a collection of Λ_i's;
(iii) for $x, y \in \iota_\ell^{-1}(\Delta_{\ell,j}),\ s_0(x,y) \geq \ell$.

To define $\mathcal{D}|\Delta_{\ell+1}$, we consider $\iota_\ell^{-1}(\Delta_{\ell,j}) \cap \{R > \ell+1\}$ and note that this set is again a union of Λ_i's. By the second requirement on $s_0(\cdot,\cdot)$ in the last subsection, the maximum number of points in Λ that are separated at or before time $\ell+1$ is finite. We partition $\iota_\ell^{-1}(\Delta_{\ell,j}) \cap \{R > \ell+1\}$ arbitrarily into finitely many sets called $\Gamma_1, \cdots, \Gamma_k$ in such a way that each Γ_j is the union of a collection of Λ_i's and for all $x, y \in \Gamma_j$, $s_0(x,y) \geq \ell+1$. The $\iota_{\ell+1}$-images of the Γ_j's will be elements of $\mathcal{D}|\Delta_{\ell+1}$.

Our construction has ensured that the image of each $\Delta_{\ell,j}$ under F is a finite union of $\Delta_{\ell+1,j'}$'s together with possibly one u-subset of Δ_0. Thus $\mathcal{D}$ is a Markov partition for $F : \Delta \circlearrowleft$ in the usual sense. Let $\Delta^*_{\ell,j} = \Delta_{\ell,j} \cap F^{-1}(\Delta_0)$. We think of $\Delta_{\ell,j} - \Delta^*_{\ell,j}$ as "moving upward" under F, while $\Delta^*_{\ell,j}$ returns to the base. Note that when $\Delta^*_{\ell,j} \neq \phi$, it is in fact a copy of one of the Λ_i's (see the fourth requirement on $s_0(\cdot,\cdot)$). Observe also that F is $1-1$ on $\Delta - \bigcup_{\ell,j} \Delta^*_{\ell,j}$, but that the images of the $\Delta^*_{\ell,j}$'s could overlap.

Next we introduce a new notion of separation time $s(\cdot,\cdot)$ defined for all pairs x, y belonging in the same $\Delta_{\ell,j}$:

$$s(x,y) \stackrel{\text{def}}{=} \text{ the largest } n \geq 0 \text{ such that for all } i \leq n,$$

$$F^i x \text{ lies in the same element of } \mathcal{D} \text{ as } F^i y$$

Note that restricted to Δ_0, $s(\cdot,\cdot) \leq s_0(\cdot,\cdot)$. Here is how I think of these two notions of separation times: $s_0(\cdot,\cdot)$ describes when two orbits in phase space genuinely cease to be comparable; this notion is natural to the dynamical system in question. On the technical level, however, it is often more convenient if "separation time $\geq n$" defines an equivalence relation, so we invent $s(\cdot,\cdot)$ which is obtained from $s_0(\cdot,\cdot)$ by artificially declaring that certain points are no longer related when in actuality they can still be compared.

Note that (P4) is valid for $x, y \in \gamma^u \cap \Delta_{\ell,j}$ with $s(\cdot,\cdot)$ in the place of $s_0(\cdot,\cdot)$. This is clearly true for $x, y \in \Lambda$ since $s(x,y) \leq s_0(x,y)$. In general, for $x, y \in \Delta_{\ell,j}$, let $x_0 = F^{-\ell}x$, $y_0 = F^{-\ell}y$ be the unique inverse images of x and y in Δ_0. Then by definition $s(x,y) = s(x_0,\ y_0) - \ell$, and (P4) is again valid for x and y.

From here on $s_0(\cdot,\cdot)$ is replaced by $s(\cdot,\cdot)$ and (P4) *is modified accordingly.*

The two views of $F : \Delta \circlearrowleft$ that we have presented can be summarized as follows. One is to regard it as a special flow over the "horseshoe" map $f^R : \Lambda \circlearrowleft$ under the return time function R. The other is to view it as the combinatorial object given by the directed graph whose vertices correspond to $\{\Delta_{\ell,j}\}$. In this graph each vertex moves upward, branching where separation occurs – except that at many vertices there is also the possibility of returning to Δ_0, the "root" of the tree.

1.4. Statements of theorems.

First we give some relevant facts and definitions.

Definition 2. An f-invariant Borel probability measure ν on M is called a *Sinai-Ruelle-Bowen* (SRB) *measure* for f if f has a positive Lyapunov exponent $\nu-$ a.e.

and the conditional measures of ν on unstable manifolds are absolutely continuous with respect to the Riemannian measures on these manifolds.

Let us restrict ourselves to systems with no zero Lyapunov exponents. If f is conservative, i.e., if it preserves a measure ν equivalent to the Riemannian measure μ on M, then from the point of view of physical observations ν is the most natural invariant measure – and it is a special case of an SRB measure. For dissipative systems, SRB measures are, in some sense, the only invariant measures that are physically observable: if ν is SRB, then there is a positive μ-measure set consisting of points that are ν-generic, i.e. $\frac{1}{n}\sum_{i=0}^{n-1}\varphi\circ f^i(x)\to\int\varphi d\nu$ for all continuous $\varphi : M\to\mathbb{R}$. (See [PS].)

If ν is equivalent to Riemannian volume or is SRB, and there are no zero Lyapunov exponents, then the phase space is decomposed into mixing components as follows: (f,ν) has at most a countable number of ergodic components supported on, say, $X_1, X_2, X_3, \ldots$; for each i, either $f : (X_i,\ \nu|X_i) \circlearrowleft$ is mixing or X_i is further decomposed into a finite cycle, i.e. $X_i = X_i^1 \cup \cdots \cup X_i^{N_i}$ with $fX_i^j = X_i^{j+1}$ and $fX_i^{N_i} = X_i^1$, and $f^{N_i} : (X_i^j,\ \nu|X_i^j) \circlearrowleft$ is mixing.

Next we turn to the speed of mixing.

Definition 3. Let ν be an f-invariant Borel probability measure and let $\mathcal{F}$ be a class of functions on M. We say that (f,ν) has *exponential decay of correlations* for functions in $\mathcal{F}$ if $\exists\tau<1$ s.t. $\forall\varphi,\psi\in\mathcal{F},\ \exists C=C(\varphi,\psi)$ s.t.

$$\left|\int(\varphi\circ f^n)\psi d\nu-\int\varphi d\nu\int\psi d\nu\right|\le C\tau^n\quad\forall n\ge 1.$$

Let (f,ν) be as above, and let $\varphi : M\to\mathbb{R}$. Consider the random variables $\varphi,\ \varphi\circ f,\ \varphi\circ f^2,\ldots$ on the probability space (M,ν). Then the exponential decay of correlations for (f,ν) says in particular that $\varphi\circ f^n$ and φ become uncorrelated exponentially fast in n. One could ask about other limit theorems.

Definition 4. Consider φ with $\int\varphi d\nu=0$. We say that φ satisfies the *Central Limit Theorem* with respect to (f,ν) if the above random variables do, i.e. if

$$\frac{1}{\sqrt{n}}\sum_{i=0}^{n-1}\varphi\circ f^i\quad\stackrel{\text{distr}}{\longrightarrow}\quad\mathcal{N}(0,\sigma)$$

for some $\sigma\ge 0$.

We now state the main results of Part I. All notations are as in 1.1-1.3, and (P1)-(P5) are assumed.

Theorem 1. *If for some $\gamma \in \Gamma^u$*

$$\int_{\gamma\cap\Lambda} R d\mu_\gamma < \infty,$$

then f admits an SRB measure.

Recall from (P5) that for all $\gamma \in \Gamma^u$, $\mu_\gamma \mid (\gamma\cap\Lambda)$ are uniformly equivalent. Hence the integrability condition above is equivalent to that on all γ^u-disks.

Let $\mathcal{H}_\eta$ denote the class of Hölder continuous functions on M with Hölder exponent η, i.e.

$$\mathcal{H}_\eta := \{\varphi : M \to \mathbb{R} \mid \exists C > 0 \text{ s.t. } |\varphi(x) - \varphi(y)| \le C d(x,y)^\eta \ \forall x, y \in M\},$$

and let ν be the SRB measure given by Theorem 1.

Theorem 2. *Suppose that*

(a) $\exists C_0 > 0$ *and* $\theta_0 < 1$ s.t. *for some* $\gamma \in \Gamma^u$,

$$\mu_\gamma\{x \in \gamma \cap \Lambda : R(x) > n\} \le C_0 \theta_0^n \quad \forall n \ge 0;$$

(b) (f^n, ν) *is ergodic* $\forall n \ge 1$.

Then (f, ν) has exponential decay of correlations for functions in $\mathcal{H}_\eta$ for every $\eta > 0$, with $\tau = \tau(\eta)$.

Theorem 3. *Under the hypotheses of Theorem 2, every $\varphi \in \mathcal{H}_\eta$ with $\int \varphi d\nu = 0$ satisfies the Central Limit Theorem* wrt (f, ν), *with $\sigma = 0$ iff $\varphi = \psi \circ f - \psi$ for some $\psi \in L^2(\nu)$.*

We prove Theorems 1-3 by working with $F : \Delta \circlearrowleft$. In particular the corresponding results hold for F. For a precise description of the class of functions on Δ to which Theorems 2 and 3 apply, see Sections 3 and 4. Theorems 1-3 also have analogs in the setting of countable state Markov chains. For example, a simplified version of our proofs gives the following result which in all likelihood is known (and in any case is not hard) but for which I have not been able to locate a reference:

Theorem. *Let $X_1, X_2, \ldots$ be a stationary Markov chain on the state space $S = \{0, 1, 2, \ldots\}$. Assume the usual ergodicity and aperiodicity conditions. Suppose also that there exist $C > 0$ and $\theta < 1$ such that for all $n > 1$,*

$$P(X_1 = 0;\ X_i \ne 0 \ \text{ for } \ i = 2, \ldots, n) < C\theta^n.$$

Then there exists $\tau < 1$ such that for all bounded $\varphi : S \to \mathbb{R}$,

$$|E(\varphi(X_1)\varphi(X_n)) - (E\varphi(X_1))^2| \le C(\varphi)\tau^n \quad \forall n \ge 1.$$

2. Sinai-Ruelle-Bowen Measures

Our proof of Theorem 1 consists of the following 3 steps:

(1) We construct an f^R-invariant finite Borel measure ν_0 on Λ with absolutely continuous conditional measures on γ^u-leaves; clearly ν_0 can be identified with an F^R-invariant measure $\tilde{\nu}_0$ on Δ_0.
(2) We extend $\tilde{\nu}_0$ to a finite F-invariant Borel measure $\tilde{\nu}$ on Δ.
(3) We verify that $\nu := \tilde{\nu}(\Delta)^{-1}\pi_*\tilde{\nu}$ has the SRB property.

To construct ν_0, we fix an arbitrary γ^u-leaf, call it γ_0, and let $\mu_0 := \mu_{\gamma_0} \mid (\gamma_0 \cap \Lambda)$. (P1) says that $\mu_0 > 0$. Let ρ_j^γ be the densities of the conditional measures of $(f^R)_*^j\mu_0$ on γ^u-leaves, i.e. $\rho_j^\gamma = \frac{d(f^R)_*^j\mu_0}{d\mu_\gamma}/((f^R)_*^j\mu_0)(\gamma)$ whenever $((f^R)_*^j\mu_0)(\gamma) > 0$. It follows from (P4)(b) that

$$\frac{\rho_j^\gamma(x)}{\rho_j^\gamma(y)} \le \exp\{C\alpha^{s(x,y)}\} \qquad \forall x, y \in \gamma \cap \Lambda.$$

In particular, $\exists M_0 > 0$ independent of j or γ s.t.

$$(*) \qquad M_0^{-1} \le \rho_j^\gamma \le M_0 \quad \text{on} \quad \gamma \cap \Lambda$$

while $\rho_j^\gamma \equiv 0$ on $\gamma - \Lambda$. Let ν_0 be an accumulation point of $\left\{\frac{1}{n}\sum_{j=0}^{n-1}(f^R)_*^j\mu_0\right\}_{n=1,2,\ldots}$ in the weak*-topology, and let $\{\nu_0^\gamma\}$ be the conditional measures of ν_0 on γ^u-leaves.

We claim that $\nu_0^\gamma \ll \mu_\gamma$ for a.e. γ with uniform bounds $M_1^{-1} \le \frac{d\nu_0^\gamma}{d\mu_\gamma} \le M_1$. To see this, fix an arbitrary open set $\omega \subset \gamma$ with $\mu_0(\partial\omega) = 0$ and let S_ω denote the s-subset of Λ corresponding to ω. Also fix a u-subset U that is a compact neighborhood of γ. Then $(*)$ together with (P5) imply that for all j,

$$(**) \qquad M_1^{-1}\frac{\mu_\gamma(S_\omega)}{\mu_\gamma(\Lambda)} \le \frac{((f^R)_*^j\mu_0)(U \cap S_\omega)}{((f^R)_*^j\mu_0)(U)} \le M_1\frac{\mu_\gamma(S_\omega)}{\mu_\gamma(\Lambda)}$$

for some M_1. (A stronger version of (P5)(b) would have allowed us to take M_1 near M_0 for U sufficiently thin.) The bounds in $(**)$ are passed on to ν_0. By taking U arbitrarily small, the martingale convergence theorem allows us to conclude that

$$M_1^{-1}\frac{\mu_\gamma(S_\omega)}{\mu_\gamma(\Lambda)} \le \nu_0^\gamma(S_\omega) \le M_1\frac{\mu_\gamma(S_\omega)}{\mu_\gamma(\Lambda)}$$

for a.e. γ. Since ω is arbitrary, the density statement for ν_0^γ follows.

For (2), let

$$\tilde{\nu} := \sum_{j=0}^{\infty} F_*^j(\tilde{\nu}_0 \mid \{R > j\}),$$

and observe that since ν_0^γ is uniformly equivalent to $\mu_\gamma \mid (\gamma \cap \Lambda)$, the finiteness of $\tilde{\nu}(\Delta)$ is equivalent to $\int_{\gamma\cap\Lambda} R d\mu_\gamma < \infty$.

As for (3), the f-invariance of ν is evident. The SRB property is also obvious since $f_*^j \nu_0$ clearly has absolutely continuous conditional measures on $\{f^j\gamma^u\}$ for every j, and these are unstable manifolds.

3. Spectral Gap of an Associated Operator

The purpose of this section is to introduce a Perron-Frobenius operator or transfer operator associated with $F : \Delta \circlearrowleft$ and to prove the existence of a spectral gap under the usual aperiodicity conditions. What exactly this tells us about stochastic processes generated by $f : (M, \nu) \circlearrowleft$ will be explained in Sections 4 and 5.

3.1. Reduction of $F : \Delta \circlearrowleft$ to an "expanding map".
Let $\bar{\Delta} := \Delta/\sim$ where $x \sim y$ iff $y \in \gamma^s(x)$. Since F takes γ^s-leaves to γ^s-leaves, the quotient dynamical system $\bar{F} : \bar{\Delta} \circlearrowleft$ is clearly well defined – topologically at least. The purpose of this subsection is to study the *differential* properties of $\bar{F}$ in the sense of the Jacobian of $\bar{F}$ with respect to a reference measure. Consider in general a measurable bijection $T : (X_1, m_1) \to (X_2, m_2)$ between two finite measure spaces. We say that T is *nonsingular* if it maps sets of m_1-measure 0 to sets of m_2-measure 0. If T is nonsingular, we define the Jacobian of T wrt m_1 and m_2, written $J_{m_1,m_2}(T)$ or simply $J(T)$, to be the Radon-Nikodym derivative $\frac{d(T_*^{-1}m_2)}{dm_1}$.

To introduce a "differential structure" for $\bar{F} : \bar{\Delta} \circlearrowleft$ in the sense above, it suffices to define a reference measure $\bar{m}$ on $\bar{\Lambda} := \Lambda/\sim$ in a way that $J_{\bar{m},\bar{m}}(\overline{f^R})$ makes sense. (Notations such as $\overline{f^R} : \overline{\Lambda} \circlearrowleft$ and $\overline{\Delta} = \underset{\ell}{\cup} \overline{\Delta}_\ell$ etc. are given the obvious meanings.) We then let $\bar{m} \mid \overline{\Delta}_\ell$ be the measure induced from the natural identification of $\overline{\Delta}_\ell$ with a subset of $\overline{\Delta}_0$, so that $J(\bar{F}) \equiv 1$ except on $\bar{F}^{-1}(\overline{\Delta}_0)$, where $J(\bar{F}) = J(\overline{f^R} \circ \bar{F}^{-(R-1)})$. We will continue to use $det(Df^u)$ to denote $J(f)$ wrt μ_γ.

We now define $\bar{m}$ on $\overline{\Lambda}$ following ideas that have been used for Axiom A (see e.g. [B]). Fix an arbitrary $\hat{\gamma} \in \Gamma^u$. For $x \in \Lambda$, let $\hat{x}$ denote the point in $\gamma^s(x) \cap \hat{\gamma}$, and define

$$u_n(x) = \sum_{i=0}^{n-1} (\varphi(f^i x) - \varphi(f^i \hat{x}))$$

where $\varphi(\cdot) = \log|\det Df^u(\cdot)|$. From (P5)(a) it follows that u_n converges uniformly to some function u. On each $\gamma \in \Gamma^u$, we let m_γ be the measure whose density wrt μ_γ is $e^u \cdot I_{\gamma\cap\Lambda}$ where $I_{(\cdot)}$ is the indicator function. Clearly, $f^{R_i} \mid (\Lambda_i \cap \gamma)$ is nonsingular wrt these reference measures. If $f^{R_i}(\Lambda_i \cap \gamma) \subset \gamma'$, then for $x \in \Lambda_i \cap \gamma$ we write $J(f^R)(x) = J_{m_\gamma, m_{\gamma'}}(f^{R_i} \mid (\Lambda_i \cap \gamma))(x)$.

Lemma 1. (1) *Let $\Theta_{\gamma,\gamma'} : \gamma \cap \Lambda \to \gamma' \cap \Lambda$ be the sliding map along Γ^s. Then $\Theta_* m_\gamma = m_{\gamma'}$.*
(2) *$J(f^R)(x) = J(f^R)(y)$ $\forall y \in \gamma^s(x)$.*

(3) $\exists C_1 > 0$ *s.t.* $\forall i$ *and* $\forall x, y \in \Lambda_i \cap \gamma$,

$$\left| \frac{J(f^R)(x)}{J(f^R)(y)} - 1 \right| \le C_1 \alpha^{\frac{1}{2}s(f^R x, f^R y)}.$$

Lemma 1 (1) allows us to define $\bar{m}$ on $\bar{\Lambda}$ to be the measure whose representative on each $\gamma \in \Gamma^u$ is m_γ. Statement (2) says that $J(f^R)$ is well defined wrt $\bar{m}$, and (3) says that $\log J(f^R)$ has a dynamically defined Hölder type property, in the sense that $\alpha^{s(f^R x, f^R y)}$ could be viewed as a notion of distance between $f^R x$ and $f^R y$ (see (P4)).

Proof of Lemma 1. (1) Suppose $\Theta x = x'$. Then the density of $\Theta_* m_\gamma$ wrt $\mu_{\gamma'}$ at x' is $e^{u(x)} \cdot \frac{d(\Theta_* \mu_\gamma)}{d\mu_{\gamma'}}$, and the second factor is $= e^{u(x') - u(x)}$ by (P5)(b).
(2) For $\mu_\gamma-$ a.e. $x \in \gamma \cap \Lambda$, we have

$$J(f^R)(x) = |\det D(f^R)^u x| \cdot e^{u(f^R x)} \cdot e^{-u(x)}.$$

We verify that $J(f^R)(x)$ depends only on $\hat{x}$ and not on x itself:

$$\begin{aligned} \log J(f^R)(x) &= \sum_{i=0}^{R-1} \varphi(f^i x) + \sum_{i=0}^{\infty} \Big(\varphi(f^i(f^R x)) - \varphi(f^i(\widehat{f^R x})) \Big) \\ &\qquad - \sum_{i=0}^{\infty} (\varphi(f^i x) - \varphi(f^i \hat{x})) \\ &= \sum_{i=0}^{R-1} \varphi(f^i \hat{x}) + \sum_{i=0}^{\infty} \Big(\varphi(f^i(f^R \hat{x})) - \varphi(f^i(\widehat{f^R x})) \Big). \end{aligned}$$

(3) We estimate $|u(x) - u(y)|$ as follows. Let $k \approx \frac{1}{2}s(x, y)$. Then

$$\begin{aligned} |u(x) - u(y)| \le &\left| \sum_{i=0}^{k-1} (\varphi(f^i x) - \varphi(f^i y)) - \sum_{i=0}^{k-1} (\varphi(f^i \hat{x}) - \varphi(f^i \hat{y})) \right| \\ &+ \left| \sum_{i=k}^{\infty} (\varphi(f^i x) - \varphi(f^i \hat{x})) - \sum_{i=k}^{\infty} (\varphi(f^i y) - \varphi(f^i \hat{y})) \right|. \end{aligned}$$

Using (P4)(b) for the first two sums and (P5)(a) for the latter two, we obtain $|u(x) - u(y)| \le 2C\alpha^{s(x,y)-k} + 2C\alpha^k \le 4C\alpha^{\frac{1}{2}s(x,y)}$. Now

$$\log \frac{J(f^R)(x)}{J(f^R)(y)} = \log \frac{\det D(f^R)^u(x)}{\det D(f^R)^u(y)} + (u(f^R x) - u(f^R y)) - (u(x) - u(y)).$$

The first term is $\le C\alpha^{s(f^R x, f^R y)}$ by (P4)(b). The second and third have been estimated above. ∎

The Perron-Frobenius operator will shall introduce will in fact be associated with $\bar{F} : \bar{\Delta} \circlearrowleft$ and will act on a suitable Banach space of functions defined on $\bar{\Delta}$. If the spectral properties of this operator are as desired, then it will have an eigenfunction $\bar{\rho}$ corresponding to the eigenvalue 1; in other words, our function space must contain an element $\bar{\rho}$ such that $\bar{\rho} d\bar{m}$ is an invariant measure for $\bar{F}$. In part to motivate the choice of this function space, we first discuss the regularity of $\bar{\rho}$. Now we have already encountered one invariant density in Section 2 obtained by collapsing $\tilde{\nu}$ along stable disks. In the next lemma we give an alternate construction.

Lemma 2. *Assume* $\int R d\bar{m} < \infty$ *and let* $\beta \geq \alpha^{\frac{1}{2}}$. *Then* $\bar{F} : \bar{\Delta} \circlearrowleft$ *has an invariant probability measure* $\bar{\nu}$ *of the form* $d\bar{\nu} = \bar{\rho} d\bar{m}$ *where* $\bar{\rho}$ *satisfies* $c_0 \leq \bar{\rho} \leq c_0^{-1}$ *for some* $c_0 > 0$ *and*

$$|\bar{\rho}(\bar{x}) - \bar{\rho}(\bar{y})| \leq C \beta^{s(\bar{x}, \bar{y})} \quad \forall \bar{x}, \bar{y} \in \bar{\Delta}_{\ell, j}.$$

Proof. We construct $\bar{\rho}$ by realizing it as the density wrt $\bar{m}$ of an accumulation point of $\bar{\nu}_n := \frac{1}{n} \sum_{i=0}^{n-1} \bar{F}_*^i(\bar{m}|\bar{\Delta}_0)$. Let us consider first $\bar{\nu}_n | \bar{\Delta}_0$. Let $\bar{\rho}_n$ be the density of $\bar{\nu}_n$ wrt $\bar{m}$. Then $\bar{\rho}_n | \bar{\Delta}_0 = \frac{1}{n} \sum_j \bar{\rho}_n^j$ where $\bar{\rho}_n^j$ is the density of $\bar{F}_*^i(\bar{m}|\sigma^j)$ and the σ^j's range over all components of $\bar{F}^{-i}(\bar{\Delta}_0) \cap \bar{\Delta}_0$, $i \leq n$. The variation of each $\bar{\rho}_n^j$ is estimated as follows: Let $\bar{x}, \bar{y} \in \bar{\Delta}_0$, and let $\bar{x}', \bar{y}' \in \sigma^j$ be s.t. $\bar{F}^i \bar{x}' = \bar{x}$, $\bar{F}^i \bar{y}' = \bar{y}$. Then

$$\frac{\bar{\rho}_n^j(\bar{y})}{\bar{\rho}_n^j(\bar{x})} = \frac{J\bar{F}^i(\bar{x}')}{J\bar{F}^i(\bar{y}')} = \prod_{k=1}^{q} \frac{J\bar{F}(\bar{F}^{i_k - 1}\bar{x}')}{J\bar{F}(\bar{F}^{i_k - 1}\bar{y}')}$$

where $i_1 < i_2 < \cdots < i_q = i$ are the times when $\bar{F}^p \sigma^j \subset \bar{\Delta}_0$, and

$$\frac{J\bar{F}(\bar{F}^{i_k - 1}\bar{x}')}{J\bar{F}(\bar{F}^{i_k - 1}\bar{y}')} \leq \exp\{C_1 \beta^{s(\bar{F}^{i_k}\bar{x}', \bar{F}^{i_k}\bar{y}')}\} \leq \exp\{C_1 \beta^{(i - i_k) + s(\bar{x}, \bar{y})}\}$$

by Lemma 1. Thus

$$\bar{\rho}_n^j(\bar{y}) \leq \bar{\rho}_n^j(\bar{x}) \cdot \exp\{C_1' \beta^{s(\bar{x}, \bar{y})}\},$$

an estimate that is easily seen to be valid also for $\bar{\rho}_n$. To finish we must let $n \to \infty$. Partitioning $\bar{\Delta}_0$ successively into sets with the property that $\bar{x}, \bar{y}$ in distinct sets satisfy $s(\bar{x}, \bar{y}) \geq k$, $k = 1, 2, \ldots$, we obtain the corresponding distortion estimate for $\bar{\rho}$.

Reasoning as in the proof of Theorem 1, we see that $\{\bar{\nu}_n\}$ has an accumulation point $\bar{\nu}$ on $\bar{\Delta}$ with $0 < \bar{\nu}(\bar{\Delta}) < \infty$. Thus we have that on $\bar{\Delta}_0, c_0 \leq \bar{\rho} \leq c_0^{-1}$ and

$$|\bar{\rho}(\bar{x}) - \bar{\rho}(\bar{y})| \leq |\bar{\rho}|_\infty \left| \frac{\bar{\rho}(\bar{x})}{\bar{\rho}(\bar{y})} - 1 \right| \leq C \beta^{s(\bar{x}, \bar{y})}.$$

For $\bar{x}, \bar{y} \in \bar{\Delta}_{\ell, j}$, use $\bar{\rho} \mid \bar{\Delta}_\ell = \bar{\rho} \circ \bar{F}^{-\ell}$. ■

We assume for the rest of Part I that $\mu_\gamma\{R \geq n\} \leq C_0 \theta_0^n$ *for every* $\gamma \in \Gamma^u$.

3.2. Choice of function space and definition of Perron-Frobenius operator.

Let $\bar F : \bar\Delta \circlearrowleft$ and $\bar m$ be as in the last subsection, and let $\{\bar\Delta_{\ell,j}\}$ be the Markov partition for $\bar F : \bar\Delta \circlearrowleft$ corresponding to $\{\Delta_{\ell,j}\}$. We collect below some important facts about $(\bar F, \bar\Delta; \bar m)$ and $\{\bar\Delta_{\ell,j}\}$. All have been introduced or proved before except for (I)(i), which is formulated precisely for the first time here. Let β be s.t. $\alpha^{\frac12} \le \beta < 1$, and let C_1 be as in Lemma 1(3).

(I) Height of tower.
 (i) $R \ge N$ *for some* N *satisfying* $C_1 e^{C_1} \beta^N \le \frac{1}{100}$;
 (ii) $\bar m\{R \ge n\} \le C_0' \theta_0^n \ \forall n \ge 0$ *for some* $C_0' > 0$ *and* $\theta_0 < 1$.

(II) Regularity of the Jacobian.
 (i) $J\bar F \equiv 1$ *on* $\bar\Delta - \bar F^{-1}(\bar\Delta_0)$,
 (ii)
$$\left|\frac{J\bar F(\bar x)}{J\bar F(\bar y)} - 1\right| \le C_1 \beta^{s(\bar F\bar x, \bar F\bar y)} \quad \forall \bar x, \bar y \in \bar\Delta^*_{\ell,j}.$$

We explain the reason for (I)(i). The exponent of β is decreased by 1 with each step up the tower; we want this gain to outdo the constants due to nonlinearities, and (I)(i) guarantees this between consecutive returns to $\bar\Delta_0$.

We now choose a function space suitable for our purposes. Let $\varepsilon > 0$ be s.t.
 (i) $e^{2\varepsilon}\theta_0 < 1$,
 (ii) $\bar m(\bar\Delta_0)^{-1} \sum\limits_{\ell,j} \bar m(\bar\Delta^*_{\ell,j}) e^{\ell\varepsilon} \le 2$.

Note that (ii) is consistent with $\sum\limits_{\ell,j} \bar m(\bar\Delta^*_{\ell,j}) = \bar m(\bar\Delta_0)$, and property (I)(ii). We remark also that β should be thought of as $< e^{-\varepsilon}$, because $\beta^N \le \frac{1}{100}$ while (ii) above implies that $e^{-\varepsilon N} \ge \frac12$.

Let $X = \{\bar\varphi : \bar\Delta \to \mathbb{C} \mid \|\varphi\| < \infty\}$ where $\|\cdot\|$ is defined as follows. We write $\bar\varphi_{\ell,j} = \bar\varphi \mid \bar\Delta_{\ell,j}$, and let $|\cdot|_p$ denote the L^p-norm wrt the reference measure $\bar m$. Then

$$\|\bar\varphi\| := \|\bar\varphi\|_\infty + \|\bar\varphi\|_h$$

where

$$\|\bar\varphi\|_\infty := \sup_{\ell,j} \|\bar\varphi_{\ell,j}\|_\infty, \quad \|\bar\varphi\|_h := \sup_{\ell,j} \|\bar\varphi_{\ell,j}\|_h,$$

and $\|\bar\varphi_{\ell,j}\|_\infty$ and $\|\bar\varphi_{\ell,j}\|_h$ are defined by

$$\begin{aligned} \|\bar\varphi_{\ell,j}\|_\infty &:= |\bar\varphi_{\ell,j}|_\infty e^{-\ell\varepsilon}, \\ \|\bar\varphi_{\ell,j}\|_h &:= \left(\operatorname*{ess\,sup}_{\bar x, \bar y \in \bar\Delta_{\ell,j}} \frac{|\bar\varphi(\bar x) - \bar\varphi(\bar y)|}{\beta^{s(\bar x, \bar y)}} \right) e^{-\ell\varepsilon}. \end{aligned}$$

It is straightforward to verify that $(X, \|\cdot\|)$ is a Banach space. Note that $\bar\rho$, the invariant density of $\bar F$, is an element of $(X, \|\cdot\|)$.

We record for future use the following relation: $\exists C_0''$ s.t. $\forall \bar\varphi \in X$,

$$|\bar\varphi|_1 \le C_0'' \|\bar\varphi\|_\infty.$$

This is true because

$$|\bar\varphi_{\ell,j}|_1 \le |\bar\varphi_{\ell,j}|_\infty \cdot \bar m(\bar\Delta_{\ell,j}) \le \|\bar\varphi_{\ell,j}\|_\infty e^{\ell\varepsilon} \bar m(\bar\Delta_{\ell,j}),$$

and so

$$|\bar\varphi|_1 \le \|\bar\varphi\|_\infty \sum_\ell \bar m(\bar\Delta_\ell) e^{\ell\varepsilon} \le \|\bar\varphi\|_\infty \sum_\ell C_0' \theta_0^\ell e^{\varepsilon\ell} < \infty.$$

The *Perron-Frobenius operator* or *transfer operator* associated with the dynamical system $\bar F : \bar\Delta \circlearrowleft$ and reference measure $\bar m$ is defined to be

$$P(\bar\varphi)(\bar x) = \sum_{\bar y : \bar F \bar y = \bar x} \frac{\bar\varphi(\bar y)}{J\bar F(\bar y)}.$$

The next few subsections are about the spectral properties of P as an operator on the function space $(X, \|\cdot\|)$.

To distinguish between $F : \Delta \circlearrowleft$ and its quotient system $\bar F : \bar\Delta \circlearrowleft$, we have, up until now, used bars $(\bar\cdot)$ to denote points, subsets and functions of the latter. *The rest of Section 3 will be exclusively about $\bar F : \bar\Delta \circlearrowleft$, and* for the sake of notational simplicity, *we will drop all the bars.*

3.3. Outline of proof of spectral gap.

Our main result is

Proposition A. (1) *P is a bounded linear operator on $(X, \|\cdot\|)$; its spectrum $\sigma(P)$ is contained in $\{|\zeta| \le 1\}$; and $\exists \tau_0 < 1$ s.t. $\sigma(P) \cap \{|\zeta| \ge \tau_0\}$ consists of a finite number of points the eigenspaces corresponding to which are all finite dimensional.*

(2) *If the greatest common divisor* (gcd) *of $\{R(z) : z \in \Delta_0\}$ is $= 1$, then 1 is the only point of $\sigma(P)$ on $\{|\zeta| = 1\}$ and it is a simple eigenvalue, i.e., its eigenspace is* 1*-dimensional.*

Our proof of (1) follows a standard route. The two main ingredients are (i) contractivity and (ii) approximation by an operator of finite rank. These two properties are made precise in Lemmas 3 and 4 below; their proofs are given in 3.4 and 3.5.

Lemma 3. (a) *$P(X) \subset X$, and $P : X \to X$ is a bounded operator.*

(b) *$\exists K > 0$ s.t. $\forall \varphi \in X$,*

$$\|P^N \varphi\| \le e^{-\varepsilon N} \|\varphi\| + K|\varphi|_1$$

where N and ε are as in 3.2.

Corollary to Lemma 3. $\sigma(P) \subset \{|\zeta| \le 1\}$.

Lemma 4. *Let τ_0 be s.t. $e^{-\varepsilon N} < \tau_0^N < 1$. Then there exists a finite rank operator $Q : X \to X$ s.t.*

$$\|P^N - Q\| < \tau_0^N.$$

Lemma 4 implies the quasi-compactness of P. (See e.g. [D&S] VIII.8.)

The main property of $F : \Delta \circlearrowleft$ used in the proof of Proposition A(2) is the following. Let ν be the F-invariant measure whose density is ρ (see e.g. Lemma 2). Then:

Lemma 5. *If $gcd\{R(z) : z \in \Delta_0\} = 1$, then (F, ν) is exact, which in this case is equivalent to (F^n, ν) being ergodic for all $n \ge 1$.*

Aside from the fact that ρ needs to be bounded away from 0, the conclusion of (2) follows from the exactness of (F, ν) and general principles not specific to the present setting. This will be explained in 3.6.

3.4. Contractivity of P.

The aim of this subsection is to prove Lemma 3 and its corollary. We distinguish between the cases $\ell \ge N$ and $\ell < n$. For $x \in \Delta_\ell$ with $\ell \ge N$, $F^{-N}\{x\}$ consists of a single point $\{y\}$ and $JF^N(y) = 1$; hence the estimates are quite trivial. On $\Delta_{\ell,j}$ with $\ell < N$, F^{-N} has infinitely many branches; they originate from distinct $\Delta_{\ell,j}$'s, and each passes through Δ_0 exactly once.

Estimate #1. For $\ell \ge N$,

$$\|(P^N\varphi)_{\ell,j}\|_\infty \le e^{-\varepsilon N}\|\varphi\|_\infty.$$

Proof.

$$\begin{aligned}
\|(P^N\varphi)_{\ell,j}\|_\infty &\overset{\text{def}}{=} |(P^N\varphi)_{\ell,j}|_\infty e^{-\ell\varepsilon} \\
&= \left(\operatorname*{ess\,sup}_{y \in F^{-N}\Delta_{\ell,j}} |\varphi y| e^{-(\ell-N)\varepsilon}\right) \cdot e^{-\varepsilon N} \\
&\le \|\varphi\|_\infty e^{-\varepsilon N}.
\end{aligned}$$

Estimate #2. $\exists K_\infty > 0$ s.t. $\forall \ell$ with $0 \le \ell < N$ and $\forall \varphi$,

$$\|(P^N\varphi)_{\ell,j}\|_\infty \le K_\infty |\varphi|_1 + 2e^{C_1}\beta^N \|\varphi\|_h.$$

Proof. We fix ℓ, j and estimate $\|\cdot\|_\infty$ by

$$(*) \qquad \|(P^N\varphi)_{\ell,j}\|_\infty \;\le\; \sum_{br} \left|\frac{1}{JF^N} I_{F^{-N}\Delta_{\ell,j}}\right|_\infty \cdot \left|\varphi I_{F^{-N}\Delta_{\ell,j}}\right|_\infty$$

where "$\sum_{br}$" means summing over all inverse branches of $F^{-N} and\ I_{(\cdot)}$ is the indicator function. We further split this sum into two sums, (1) and (2), corresponding to estimating each branch of $|\varphi I_{F^{-N}\Delta_{\ell,j}}|_\infty$ by

$$\left|\varphi I_{F^{-N}\Delta_{\ell,j}}\right|_\infty \le \left|\frac{1}{m(F^{-N}\Delta_{\ell,j})}\int_{F^{-N}\Delta_{\ell,j}} \varphi dm\right| + \operatorname*{ess\ sup}_{y_1,y_2\in F^{-N}\Delta_{\ell,j}} |\varphi y_1 - \varphi y_2|.$$

Using the distortion estimate

$$\left|\frac{1}{JF^N} I_{F^{-N}\Delta_{\ell,j}}\right|_\infty \le e^{C_1}\cdot \frac{m(F^{-N}\Delta_{\ell,j})}{m(\Delta_{\ell,j})},$$

we obtain that

$$(1) \le \sum_{br} \frac{e^{C_1}}{m(\Delta_{\ell,j})}\cdot \left|\varphi I_{F^{-N}\Delta_{\ell,j}}\right|_1 \le K_\infty |\varphi|_1$$

where $K_\infty := e^{C_1}\cdot \max\{m(\Delta_{\ell',j'})^{-1} : \ell' < N,\ \text{all } j'\}$ is finite because there are only finitely many $\Delta_{\ell',j'}$'s with $\ell' < N$. To estimate (2), let ℓ_{br} be the level of the branch of $F^{-N}\Delta_{\ell,j}$ in question. Let $\Delta^*_{br} = \Delta^*_{\ell',j'}$, where $\Delta_{\ell',j'}$ is the component on the top level of Δ through which this branch passes, and let $\ell^*_{br} = \ell'$. Then

$$(2) \le \left[\sum_{br}\left|\frac{1}{JF^N} I_{F^{-N}\Delta_{\ell,j}}\right|_\infty \cdot \left(\operatorname*{ess\ sup}_{y_1,y_2\in F^{-N}\Delta_{\ell,j}} \frac{|\varphi y_1 - \varphi y_2|}{\beta^{s(y_1,y_2)}} e^{-\ell_{br}\varepsilon}\right)\cdot e^{\ell_{br}\varepsilon}\right]\beta^N$$

(**)

$$\le e^{C_1}\|\varphi\|_h \beta^N \cdot \left(\frac{1}{m\Delta_0}\sum_{br} m\Delta^*_{br} e^{\ell^*_{br}\varepsilon}\right)$$

In the first inequality above we have used the fact that $\forall y_1, y_2 \in F^{-N}\Delta_{\ell,j},\ s(y_1,y_2) \ge N$, and in the second we have used the distortion estimate

$$\left|\frac{1}{JF^N} I_{F^{-N}\Delta_{\ell,j}}\right|_\infty \le e^{C_1}\frac{m(\Delta^*_{br})}{m(\Delta_0)}.$$

The quantity in parenthesis in (**) is clearly ≤ 2.

Estimate #3. For $\ell \ge N$,

$$\|(P^N\varphi)_{\ell,j}\|_h \le \beta^N e^{-\varepsilon N}\|\varphi\|_h.$$

Proof.

$$\begin{aligned}\|(P^N\varphi)_{\ell,j}\|_h &\stackrel{\text{def}}{=} \left(\operatorname*{ess\ sup}_{x_1,x_2\in\Delta_{\ell,j}} \frac{|(P^N\varphi)x_1 - (P^N\varphi)x_2|}{\beta^{s(x_1,x_2)}}\right)\cdot e^{-\ell\varepsilon}\\ &= \left(\operatorname*{ess\ sup}_{y_1,y_2\in F^{-N}\Delta_{\ell,j}} \frac{|\varphi y_1 - \varphi y_2|}{\beta^{s(y_1,y_2)}}\cdot e^{-(\ell-N)\varepsilon}\right)\beta^N e^{-\varepsilon N}\\ &\le \|\varphi\|_h \beta^N e^{-\varepsilon N}.\end{aligned}$$

Estimate #4. For $\ell < N$,

$$\|(P^N\varphi)_{\ell,j}\|_h \le C_1 K_\infty |\varphi|_1 + 4C_1 e^{C_1}\beta^N \|\varphi\|_h.$$

Proof. Writing $y_i = F^{-N}x_i$ for $x_1, x_2 \in \Delta_{\ell,j}$, we have

$$\begin{aligned}
\|(P^N\varphi)_{\ell,j}\|_h &\stackrel{\text{def}}{=} \operatorname*{ess\,sup}_{x_1,x_2\in\Delta_{\ell,j}} \left(\left| \sum_{br} \left(\frac{\varphi y_1}{JF^N y_1} - \frac{\varphi y_2}{JF^N y_2} \right) \right| \beta^{-s(x_1,x_2)} \right) \cdot e^{-\ell\varepsilon} \\
(***) \qquad &\le \sum_{br} \operatorname*{ess\,sup}_{y_1,y_2\in F^{-N}\Delta_{\ell,j}} \left(\left| \frac{\varphi y_1}{JF^N y_1} - \frac{\varphi y_2}{JF^N y_2} \right| \beta^{-s(y_1,y_2)} \right) \cdot \beta^N
\end{aligned}$$

For each inverse branch,

$$\begin{aligned}
\left| \frac{\varphi y_1}{JF^N y_1} - \frac{\varphi y_2}{JF^N y_2} \right| &\le \frac{|\varphi y_1 - \varphi y_2|}{JF^N y_1} + |\varphi y_2| \left| \frac{1}{JF^N y_1} - \frac{1}{JF^N y_2} \right| \\
&\le \frac{|\varphi y_1 - \varphi y_2|}{JF^N y_1} + \frac{|\varphi y_2|}{JF^N y_2} \cdot C_1 \beta^{s(F^{N-\ell}y_1, F^{N-\ell}y_2)}.
\end{aligned}$$

(In the last line we have used the fact that $F^{N-\ell}y_i \in \Delta_0$, $JF^N y_i = JF(F^{N-\ell-1}y_i)$, and the distortion estimate for JF.) We may now write $(***) \le (3) + (4)$ where

$$(3) := \sum_{br} \left| \frac{1}{JF^N} I_{F^{-N}\Delta_{\ell,j}} \right|_\infty \cdot \left(\operatorname*{ess\,sup}_{y_1,y_2\in F^{-N}\Delta_{\ell,j}} \frac{|\varphi y_1 - \varphi y_2|}{\beta^{s(y_1,y_2)}} \right) \cdot \beta^N$$

and

$$(4) := C_1 \sum_{br} \left| \frac{1}{JF^N} I_{F^{-N}\Delta_{\ell,j}} \right|_\infty \cdot \left| \varphi I_{F^{-N}\Delta_{\ell,j}} \right|_\infty \cdot \beta^N.$$

Observe that (3) is exactly the line above (**) in *Estimate #2*, and (4) differs only by a constant from the right side of the inequality in (*) in the same estimate.

Putting these 4 estimates together and recalling that $e^{-\varepsilon N} > \beta^N$, we conclude that

$$\begin{aligned}
\|P^N\varphi\| &\le (1 + C_1)K_\infty |\varphi|_1 + e^{-\varepsilon N}\|\varphi\|_\infty + 10C_1 e^{C_1}\beta^N \|\varphi\|_h \\
&\le K|\varphi|_1 + e^{-\varepsilon N}\|\varphi\|
\end{aligned}$$

for some K completing the proof of Lemma 3. ■

Proof of Corollary to Lemma 3. To prove that the spectrum of P lies in the closed unit disk, it suffices to show

$$\sup_n \|P^n\| < \infty.$$

Using the fact that $|P\varphi|_1 \le |\varphi|_1$, we have for all $k \in \mathbb{Z}^+$ and $\varphi \in X$,

$$\begin{aligned}\|P^{kN}\varphi\| \le\ & e^{-\varepsilon N}\|P^{(k-1)N}\varphi\| + K|P^{(k-1)N}\varphi|_1 \\ & \vdots \\ \le\ & e^{-\varepsilon kN}\|\varphi\| + K\left(\sum_j e^{-\varepsilon jN}\right)|\varphi|_1.\end{aligned}$$

Since $|\varphi|_1 \le C_0''\|\varphi\|_\infty$ (see 3.2), we have shown that $\|P^{kN}\varphi\| \le K_0\|\varphi\|$ for some K_0. For $n = kN + i,\ i < N$, we have $\|P^n\varphi\| \le K_0\|P^i\varphi\| \le K_0\|P\|^N\|\varphi\|$. ■

3.5. Approximation of P by an operator of finite rank.

The aim of this subsection is to prove Lemma 4. Let $\mathcal{M}_0$ be the partition of Δ into $\Delta_{\ell,j}$-components (i.e. $\mathcal{M}_0 = \mathcal{D}$ in 1.3), and let $\mathcal{M}_N = \bigvee_0^N F^{-i}\mathcal{M}_0$. For $\varphi : \Delta \to \mathbb{C}$, we let $E_N(\varphi)$ denote the expectation of φ wrt m on elements of $\mathcal{M}_N$. For $k \in \mathbb{Z}^+$, we define $\varphi^{\le k} := \varphi I_{\bigcup\limits_{\ell\le k}\Delta_\ell}$; similarly, $\varphi^{>k} := \varphi I_{\bigcup\limits_{\ell>k}\Delta_\ell}$. Consider the operator Q_k on X defined by

$$Q_k(\varphi) = P^N(E_N(\varphi^{\le k})).$$

Since the number of components on each level is finite, it is evident that Q_k has finite rank. We will show that $\|P^N - Q_k\| < \tau_0^N$ for all sufficiently large k.

In the discussion to follow it is convenient to write

$$(P^N - Q_k)(\varphi) = P^N(\psi) + P^N(\varphi^{>k})$$

where $\psi = (\varphi - E_N(\varphi))^{\le k}$. Note that $E_N(\psi) \equiv 0$.

Estimate #5. $\|P^N\psi\| \le \frac{1}{10}\|\varphi\|_h$.

Proof. As before there are 4 cases to consider: $\ell \ge N$ and $\ell < N,\ \|\cdot\|_\infty$ and $\|\cdot\|_h$. The $\|\cdot\|_\infty$ term for $\ell \ge N$ is dealt with a little differently than before:

$$\begin{aligned}\|(P^N\psi)_{\ell,j}\|_\infty \overset{\text{def}}{=}\ & |\psi I_{F^{-N}\Delta_{\ell,j}}|_\infty \cdot e^{-\ell\varepsilon} \\ \le\ & |\mathrm{Avg}(\psi \mid F^{-N}\Delta_{\ell,j})| + \operatorname*{ess\,sup}_{y_1,y_2\in F^{-N}\Delta_{\ell,j}} |\psi y_1 - \psi y_2|.\end{aligned}$$

Here, $\mathrm{Avg}(\psi \mid F^{-N}\Delta_{\ell,j}) = 0$ and $|\psi y_1 - \psi y_2| = |\varphi y_1 - \varphi y_2|$ assuming $\ell - N \le k$ (otherwise there is nothing to prove). This gives $\|(P^N\psi)_{\ell,j}\|_\infty \le \beta^N e^{-\varepsilon N}\|\varphi\|_h$.

The other 3 cases follow closely their counterparts in 3.4 except that the $|\cdot|_1$-terms are absent because $E_N(\psi) \equiv 0$. Note that as expected, $\|\varphi\|_\infty$ does not appear in the estimate.

Estimate #6. $\exists \varepsilon_k$ with $\varepsilon_k \to 0$ as $k \to \infty$ s.t.

$$\|P^N(\varphi^{>k})\| \leq (e^{-\varepsilon N} + \varepsilon_k)\|\varphi\|_\infty + \frac{1}{10}\|\varphi\|_h.$$

Proof. The estimates for $\ell \geq N$ are identical with those in *Estimates #1* and *3* and we omit them. Consider the $\|\cdot\|_\infty$-norm for $\ell < N$. Let $\sum\limits_{br}^{>k}$ denote the sum over all inverse branches with $\ell_{br} > k$. Then

$$\begin{aligned}
\|(P^N(\varphi^{>k})_{\ell,j})\|_\infty &\leq \sum_{br}{}^{>k} \left|\frac{1}{JF^N} I_{F^{-N}\Delta_{\ell,j}}\right|_\infty \cdot \left|\varphi I_{F^{-N}\Delta_{\ell,j}}\right|_\infty \\
&\leq \sum_{br}{}^{>k} e^{C_1} \frac{m(F^{-N}\Delta_{\ell,j})}{m(\Delta_{\ell,j})} \cdot \|\varphi\|_\infty e^{\ell_{br}\varepsilon} \\
&\leq \frac{e^{C_1}}{m(\Delta_{\ell,j})}\|\varphi\|_\infty \sum_{\ell>k} m(\Delta_\ell) e^{\ell\varepsilon}
\end{aligned}$$

which is $< \varepsilon_k' \|\varphi\|_\infty$ for some ε_k' with $\varepsilon_k' \to 0$ as $k \to \infty$. The $\|\cdot\|_h$-norm for $\ell < N$ is dealt with as in *Estimate #4*; part of it refers back to the above estimate.

Choosing k s.t. $\varepsilon_k + e^{-\varepsilon N} < \tau_0^N$ and remembering that $e^{-\varepsilon N} > \frac{1}{2}$, we have proved that for all $\varphi \in X$,

$$\|(P^N - Q_k)\varphi\| < \frac{1}{5}\|\varphi\|_h + \tau_0^N \|\varphi\|_\infty \leq \tau_0^N \|\varphi\|.$$

∎

We remark that Lemmas 3 and 4 are valid even when the number of $\Delta_{\ell,j}$'s on each level is not assumed to be finite for $\ell \geq N$. (For $\ell < N$, this finiteness is used in a rather essential way in Estimate #2.) For $\ell \geq N$, it is used only to ensure that Q_k has finite rank. This can be modified as follows. For each k, define $\varphi^{\leq k} : -\varphi I_{\Sigma_k}$ where Σ_k is a union of finitely many components of $\Delta_{\ell,j}$ chosen in such a way that

$$\sum_{\Delta_{\ell,j} \not\subset \Sigma_k} m(\Delta_{\ell,j}) e^{\ell\varepsilon} \to 0 \quad \text{as} \quad k \to \infty.$$

We may then define $\varphi^{>k} := \varphi - \varphi^{\leq k}$ and proceed as before.

3.6. Ruling out other eigenvalues of modulus 1.

Let $\mathcal{B}$ be the σ-algebra of Borel subsets of Δ. Recall that (F, ν) is called *exact* if $\bigcap\limits_{n\geq 0} F^{-n}\mathcal{B}$ is trivial in the sense that it contains only sets having ν-measure 0 or 1. We begin with the following observation:

Sublemma. *Suppose that* $gcd\{R(z)\} = 1$. *Then for every* $\ell_0 \in \mathbb{Z}^+$, $\exists t_0 \in \mathbb{Z}^+$ *s.t.* $F^{t_0}(\Delta_0) \supset \bigcup_{\ell \le \ell_0} \Delta_\ell$.

Proof. Because of the nature of Markov partitions, it suffices to produce t_0' s.t. $m(\Delta_0 \cap F^{-t}\Delta_0) > 0\ \forall t \ge t_0'$, for then we could take $t_0 = t_0' + \ell_0$. The existence of t_0' follows from the gcd assumption as in the proof of $p_{ij}^{(n)} > 0$ for irreducible aperiodic finite state Markov chains. ∎

Proof of Lemma 5. We prove the exactness of (F, ν). Let $A \in \bigcap_{n \ge 0} F^{-n}\mathcal{B}$ be s.t. $\nu(A) > 0$. It suffices to show that $\nu(A) > 1 - \varepsilon_1$ for every $\varepsilon_1 > 0$. From the Sublemma it follows that $\exists t_1 = t_1(\varepsilon_1) \in \mathbb{Z}^+$ and $\delta_1 = \delta_1(\varepsilon_1, t_1) > 0$ s.t. if $B \in \mathcal{B}$ satisfies $m(\Delta_0 - B) < \delta_1$, then $\nu(f^{t_1}B) > 1 - \varepsilon_1$. We claim that it suffices to show $m(\Delta_0 - F^n A) < \delta_1$ for some $n > 0$. Assuming this, we have, since $A \in F^{-(n+t_1)}\mathcal{B}$, that $A = F^{-(n+t_1)}A'$ for some $A' \in \mathcal{B}$. Hence $\nu(A) = \nu(A') = \nu(F^{t_1}(F^n A)) > 1 - \varepsilon_1$.

To produce an n with the property above, let $\Delta_{\ell,j}$ be s.t. $m(A \cap \Delta_{\ell,j}) > 0$, and consider the increasing σ-algebra on $\Delta_{\ell,j}$ defined by $\mathcal{M}_k$, $k = 1, 2, \ldots$ (see 3.5 for definition). Clearly, $E(I_A \mid \mathcal{M}_k) \to I_A$ m-a.s. as $k \to \infty$. Pick a typical point $x \in A \cap \Delta_{\ell,j}$ and choose a sufficiently large n s.t. $F^n(\mathcal{M}_n(x)) = \Delta_0$. Then our distortion estimate for m (see Lemma 2) gives

$$\frac{mF^n(A \cap \mathcal{M}_n(x))}{m(\Delta_0)} \approx \frac{m(A \cap \mathcal{M}_n(x))}{m(\mathcal{M}_n(x))} \approx 1$$

as required. ∎

Proof of Proposition A(2). We finish by explaining how the exactness of (F, ν) implies the conclusion of Proposition A(2). Let $\varphi \in X$, $\varphi \not\equiv 0$, be s.t. $P(\varphi) = \xi\varphi$ for some $\xi \in \mathbb{C}$ with $|\xi| = 1$. We write $\varphi = \theta\rho$, which is legitimate since $\rho \ge c_0 > 0$ (Lemma 2), and observe that $\theta \in L^2(m)$ because $|\theta_{\ell,j}|_\infty \le \frac{\|\varphi\|}{c_0} e^{\varepsilon\ell}$ and $\sum_\ell e^{2\ell\varepsilon} m(\Delta_\ell) < \infty$. With m uniformly equivalent to ν, this puts $\theta \in L^2(\nu)$. Consider $U : L^2(\nu) \to L^2(\nu)$ defined by $U(\psi) = \psi \circ F$ and let U^* be the adjoint of U. Then $U^*(\theta) = \xi\theta$ because $\int (U^*\theta)\bar{\psi} d\nu = \int \theta(\overline{U\psi}) d\nu = \int (\bar{\psi} \cdot F)\theta\rho dm = \int \bar{\psi} P(\theta\rho) dm = \int \left(\frac{P(\theta\rho)}{\rho}\right) \bar{\psi} d\nu$ for all $\psi \in L^2(\nu)$. It follows from this that $\theta = \xi U(\theta)$, hence $\theta = \xi^n U^n(\theta)\ \forall n \ge 1$, which implies that θ is measurable wrt $F^{-n}\mathcal{B}\ \forall n \ge 1$. The exactness of (F, ν) then tells us that $\theta \equiv$ const a.e. This in turn forces $\xi = 1$ because $\theta P(\rho) = P(\theta\rho) = \theta\xi\rho$ (and $\theta \ne 0$), proving that 1 is the only spectral point of P with modulus 1. Let X_1 be the image of the projection associated with $1 \in \sigma(P)$. We have shown that $\dim X_1 < \infty$. The number of Jordan blocks for $P \mid X_1$ cannot exceed 1 because ρ is the unique invariant density of F, and there can be no subdiagonal 1's in the block because that would contradict $\sup_n \|P^n\| < \infty$. ∎

We remark that there are other standard ways of dealing with eigenvalues on the unit circle. One could, for instance, argue that for positive operators like P, all eigenvalues of modulus 1 are n^{th} roots of unity, and for P their eigenfunctions are invariant densities of F^n. The problem then boils down once again to proving the ergodicity of (F^n, ν) for all $n \geq 1$ as we have shown.

4. Exponential Decay of Correlations

Recall that we have constructed a Markov extension $F : (\Delta,\ \tilde{\nu}) \circlearrowleft$ over our dynamical system of interest $f : (M,\ \nu) \circlearrowleft$ where ν is an SRB measure and $\pi : \Delta \to M$ sends $\tilde{\nu}$ to ν. We have also constructed a quotient system $\overline{F} : (\overline{\Delta},\ \overline{\nu}) \circlearrowleft$ by collapsing γ^s-leaves in $F : (\Delta,\ \tilde{\nu}) \circlearrowleft$, and the projection map $\overline{\pi} : \Delta \to \overline{\Delta}$ sends $\tilde{\nu}$ to $\overline{\nu}$.

Let us use the following convention: if φ is a function on M, then $\tilde{\varphi}$ is the lift of φ to Δ, i.e. $\tilde{\varphi} = \varphi \circ \pi$; and if $\tilde{\varphi}$ is constant on γ^s-leaves, then we will confuse it with the function on $\overline{\Delta}$ called $\overline{\varphi}$. (It would have been logical to write $\tilde{F} : (\tilde{\Delta},\ \tilde{\nu}) \circlearrowleft$ instead of $F : (\Delta,\ \tilde{\nu}) \circlearrowleft$ but let us not do that.)

We assume for the rest of Part I that $(f^n,\ \nu)$ is ergodic for all $n \geq 1$.

4.1. Reduction of problem.

The purpose of this subsection is to reformulate the problem of decay of correlations for $(f,\ \nu)$ in a way that the properties of the Perron-Frobenius operator studied in Section 3 can be brought to bear. Recall that

$$\mathcal{H}_\eta := \{\varphi : M \to \mathbb{R} \mid \exists C = C_\varphi \text{ s.t. } \forall x, y \in M,\ |\varphi x - \varphi y| \leq C d(x,y)^\eta\}.$$

We will use $D_n(\varphi, \psi; \nu)$ to denote the correlation between φ and $\psi \circ f^n$ wrt the probability measure ν, i.e.

$$D_n(\varphi, \psi; \nu) := \left| \int (\psi \circ f^n) \varphi d\nu - \int \varphi d\nu \int \psi d\nu \right|.$$

Our first observation is that to prove Theorem 2, it suffices to prove the corresponding result for $F : (\Delta, \tilde{\nu}) \circlearrowleft$ and test functions of the form $\tilde{\varphi}, \tilde{\psi}$ with $\varphi, \psi \in \mathcal{H}_\eta$. This is because $D_n(\varphi, \psi; \nu) = D_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu})$, which is straightforward to verify. Next we observe that we may assume gcd $\{R(x) : x \in \Delta_0\} = 1$. Suppose not. Let $N_1 = \gcd\{R\}$. Instead of $F : (\Delta, \tilde{\nu}) \circlearrowleft$, we will consider $F^{N_1} : \left(\Delta^{(N_1)}, \tilde{\nu}^{(N_1)}\right) \circlearrowleft$ where $\Delta^{(N_1)} := \bigcup_{k=0}^{\infty} \Delta_{kN_1}$ and $\tilde{\nu}^{(N_1)} := \tilde{\nu} \mid \Delta^{(N_1)}$ normalized. Since $\pi_* \tilde{\nu}^{(N_1)}$ is an f^{N_1}-invariant probability measure on M and it is absolutely continuous wrt ν, it follows from the ergodicity of (f^{N_1}, ν) that $\pi_* \tilde{\nu}^{(N_1)} = \nu$. Suppose we have the desired result for $F^{N_1} : \left(\Delta^{(N_1)}, \tilde{\nu}^{(N_1)}\right) \circlearrowleft$ and hence for $f^{N_1} : (M, \nu) \circlearrowleft$. For given $\varphi, \psi \in \mathcal{H}_\eta$, exponential decay of $D_{nN_1}(\varphi, \psi \circ f^i; \nu)$ for $i = 0, 1, \ldots, N_1 - 1$ clearly implies that of $D_n(\varphi, \psi; \nu)$. From now on we assume gcd $\{R\} = 1$.

The following geometric fact about the "sizes" of the $\Delta_{\ell,j}$'s is used to relate the Hölder property of functions in $\mathcal{H}_\eta$ to a corresponding property for their lifts to Δ:

Sublemma. $\forall x \in \Delta,\ \operatorname{diam}(\pi F^k(\mathcal{M}_{2k}(x))) \le 2C\alpha^k$.

Proof. Let $y_1, y_2 \in \mathcal{M}_{2k}(x)$. Then $\exists \hat{y} \in \gamma^u(y_1) \cap \gamma^s(y_2)$. Suppose $\mathcal{M}_{2k}(x) \subset \Delta_\ell$. Then $\pi F^{-\ell}\hat{y}, \pi F^{-\ell}y_2$ are both in Δ_0 and they lie on the same γ^s-leaf. By (P3), we have $d(\pi F^k\hat{y}, \pi F^k y_2) < C\alpha^{\ell+k}$. Applying (P4)(a) to $\hat{y}$ and y_1 and noting that $s(F^k\hat{y}, F^k y_1) \ge k$, we obtain that $d(\pi F^k\hat{y}, \pi F^k y_1) < C\alpha^k$. ■

We now proceed to argue that $D_n(\tilde{\varphi}, \tilde{\psi}; \tilde{\nu})$ can be approximated by a quantity that involves only $\overline{F} : (\overline{\Delta}, \overline{\nu}) \circlearrowleft$ and functions on $\overline{\Delta}$, and that the error decreases exponentially with n. For each $n \in \mathbb{Z}^+$, let $k = k(n)$ be a number $< \frac{1}{2}n$ to be determined.

Approximation #1. Define $\overline{\psi}_k$ on Δ (or $\overline{\Delta}$) by $\overline{\psi}_k \mid A = \inf\{\psi(x) : x \in F^k A\}$ for every $A \in \mathcal{M}_{2k}$. Then

$$|D_{n-k}(\tilde{\varphi}, \tilde{\psi} \circ F^k; \tilde{\nu}) - D_{n-k}(\tilde{\varphi}, \overline{\psi}_k; \tilde{\nu})| \le C'\alpha^{k\eta}$$

for some $C' = C'(\varphi, \psi)$.

Proof. It follows from the Sublemma that $|\tilde{\psi} \circ F^k - \overline{\psi}_k| \le C_\psi(2C\alpha^k)^\eta$. Hence the quantity in question is

$$\begin{aligned}
&\le \left| \int (\tilde{\psi} \circ F^k - \overline{\psi}_k) \circ F^{n-k} \cdot \tilde{\varphi} d\tilde{\nu} \right| + \left| \int (\tilde{\psi} \circ F^k - \overline{\psi}_k) d\tilde{\nu} \cdot \int \tilde{\varphi} d\tilde{\nu} \right| \\
&\le 2C_\psi(2C\alpha^k)^\eta \cdot \max|\varphi| \le C'\alpha^{k\eta}.
\end{aligned}$$

■

The next 2 steps are made slightly more complicated by the fact that F need not be one-to-one.

Approximation #2. Let $\overline{\psi}_k$ be as above, and let $\overline{\varphi}_k$ be defined analogously. Let $\overline{\varphi}_k\tilde{\nu}$ denote the signed measure whose density wrt $\tilde{\nu}$ is $\overline{\varphi}_k$, and let $\tilde{\varphi}_k := d(F^k_*(\overline{\varphi}_k\tilde{\nu}))/d\tilde{\nu}$. Then

$$\left|D_{n-k}(\tilde{\varphi}, \overline{\psi}_k; \tilde{\nu}) - D_{n-k}(\tilde{\varphi}_k, \overline{\psi}_k; \tilde{\nu})\right| \le C''\alpha^{k\eta}$$

for some $C'' = C''(\varphi, \psi)$.

Proof. The quantity in question is

$$\begin{aligned}
&\le \left| \int (\overline{\psi}_k \circ F^{n-k})(\tilde{\varphi} - \tilde{\varphi}_k) d\tilde{\nu} \right| + \left| \int \overline{\psi}_k d\tilde{\nu} \cdot \int (\tilde{\varphi} - \tilde{\varphi}_k) d\tilde{\nu} \right| \\
&\le (2\max|\psi|) \cdot \int |\tilde{\varphi} - \tilde{\varphi}_k| d\tilde{\nu}.
\end{aligned}$$

Letting $|\cdot|$ denote the total variation of a signed measure, and noting that $F_*^k((\tilde{\varphi}\circ F^k)\tilde{\nu}) = \tilde{\varphi}\tilde{\nu}$, we have

$$\int |\tilde{\varphi}-\tilde{\varphi}_k|d\tilde{\nu} = |\tilde{\varphi}\tilde{\nu}-\tilde{\varphi}_k\tilde{\nu}| = \left|F_*^k((\tilde{\varphi}\circ F^k)\tilde{\nu}) - F_*^k(\overline{\varphi}_k\tilde{\nu})\right|$$
$$\leq \left|(\tilde{\varphi}\circ F^k - \overline{\varphi}_k)\tilde{\nu}\right| = \int |\tilde{\varphi}\circ F^k - \overline{\varphi}_k| d\tilde{\nu},$$

and this last quantity has been estimated above. ■

Finally, we verify that $D_{n-k}(\tilde{\varphi}_k, \overline{\psi}_k; \tilde{\nu})$ can be expressed purely in terms of objects related only to $\overline{F} : (\overline{\Delta}, \overline{\nu}) \circlearrowleft$. First,

$$\int (\overline{\psi}_k \circ F^{n-k})\tilde{\varphi}_k d\tilde{\nu} = \int \overline{\psi}_k d(F_*^{n-k}(\tilde{\varphi}_k\tilde{\nu})) = \int \overline{\psi}_k d(F_*^n(\overline{\varphi}_k\tilde{\nu})),$$

and since $\overline{\psi}_k$ is constant on γ^s and F commutes with $\overline{\pi}$, we have

$$\int \overline{\psi}_k d(F_*^n(\overline{\varphi}_k\tilde{\nu})) = \int \overline{\psi}_k d(\overline{\pi}_* F_*^n(\overline{\varphi}_k\tilde{\nu})) = \int \overline{\psi}_k d(\overline{F}_*^n(\overline{\varphi}_k\overline{\nu})),$$

which, in the language of Section 3, is equal to

$$\int \overline{\psi}_k P^n(\overline{\varphi}_k\overline{\rho}) d\overline{m}.$$

Also,

$$\int \tilde{\varphi}_k d\tilde{\nu} \cdot \int \overline{\psi}_k d\tilde{\nu} = \int d(F_*^k(\overline{\varphi}_k\tilde{\nu})) \cdot \int \overline{\psi}_k d\overline{\nu} = \int \overline{\varphi}_k d\overline{\nu} \cdot \int \overline{\psi}_k d\overline{\nu}.$$

4.2. Estimating $D_{n-k}(\tilde{\varphi}_k, \overline{\psi}_k; \tilde{\nu})$ and completing the proof.
¿From the last subsection we have

$$D_{n-k}(\tilde{\varphi}_k, \overline{\psi}_k; \tilde{\nu}) = \left|\int \overline{\psi}_k \left\{P^n(\overline{\varphi}_k\overline{\rho}) - \left(\int \overline{\psi}_k \overline{\rho} d\overline{m}\right)\overline{\rho}\right\} d\overline{m}\right|$$
$$\leq \max|\psi| \cdot \left|P^n(\overline{\varphi}_k\overline{\rho}) - \left(\int \overline{\varphi}_k\overline{\rho} d\overline{m}\right)\overline{\rho}\right|_1$$
$$\leq \max|\psi| \cdot C_0'' \left\|P^n(\overline{\varphi}_k\overline{\rho}) - \left(\int \overline{\varphi}_k\overline{\rho} d\overline{m}\right)\overline{\rho}\right\|$$

where $\|\cdot\|$ is the norm introduced in 3.2.

Sublemma. *$\exists \overline{C}_\varphi$ depending only on φ s.t. $\|P^{2k}(\overline{\varphi}_k\overline{\rho})\| \le \overline{C}_\varphi$ for all k.*

Proof. On each $\overline{\Delta}_{\ell,j}, \overline{\rho} = P^{2k}\overline{\rho}$ can be written as $\sum\limits_i \overline{\rho}^i_{2k}$ where $\overline{\rho}^i_{2k}$ is the contribution from each of the inverse branches of $\overline{F}^{2k}$. Since $\overline{\varphi}_k$ is constant on elements of $\mathcal{M}_{2k}, P^{2k}(\overline{\varphi}_k\overline{\rho})$ on each $\overline{\Delta}_{\ell,j}$ has the form $\sum\limits_i c_i\overline{\rho}^i_{2k}$, and

$$\begin{aligned} |P^{2k}(\overline{\varphi}_k\overline{\rho})(\overline{x}) - P^{2k}(\overline{\varphi}_k\overline{\rho})(\overline{y})| \le & \sum_i |c_i||\overline{\rho}^i_{2k}(\overline{x}) - \overline{\rho}^i_{2k}(\overline{y})| \\ \le & \max|\varphi| \cdot \overline{\rho}(\overline{x}) \cdot C\beta^{s(\overline{x},\overline{y})} \end{aligned}$$

as in the proof of lemma 2. ■

Let $\tau_1 > \sup\{|\zeta| : \zeta \in \sigma(P), \zeta \ne 1\}$. The sublemma above suggests that we write

$$\left\| P^n(\overline{\varphi}_k\overline{\rho}) - \left(\int \overline{\varphi}_k\overline{\rho}d\overline{m}\right)\overline{\rho}\right\| = \left\| P^{n-2k}(P^{2k}(\overline{\varphi}_k\overline{\rho})) - \left(\int P^{2k}(\overline{\varphi}_k\overline{\rho})d\overline{m}\right)\overline{\rho}\right\|.$$

Since for $h \in X$, $\left(\int hd\overline{m}\right)\overline{\rho}$ is the projection of h onto the $1-d$ subspace spanned by $\overline{\rho}$, the quantity above is

$$\begin{aligned} &\le C\tau_1^{n-2k}\left\| P^{2k}(\overline{\varphi}_k\overline{\rho}) - \left(\int P^{2k}(\overline{\varphi}_k\overline{\rho})d\overline{m}\right)\overline{\rho}\right\| \\ &\le \text{ const } \cdot \tau_1^{n-2k}. \end{aligned}$$

Combining the arguments in the last two subsections, we see that if for instance we let $k = \kappa n$, $\kappa \in \left(0, \frac{1}{2}\right)$, then we have for all $\varphi, \psi \in \mathcal{H}_\eta$,

$$D_n(\varphi, \psi; \nu) \le C\tau^n$$

for $\tau = \max\{\alpha^{\kappa\eta}, \tau_1^{(1-2\kappa)}\}$. ■

5. Central Limit Theorem

5.1. CLT for dynamical systems: background information.

In this subsection we review three known results related to the asymptotic normality of random variables generated from dynamical systems. Notations in 5.1 are independent of those in the rest of this paper.

A. Conditions for CLT for measure-preserving transformations: a theorem of Gordin.

Theorem [G]. Invertible case. *Let $T : (\Omega, \mathcal{M}, \nu) \circlearrowleft$ be an invertible measure preserving transformation of a probability space, and let $\varphi \in L^2(\nu)$ be s.t. $E\varphi = 0$. Suppose there exists a sub-σ-algebra $\mathcal{M}_0 \subset \mathcal{M}$ s.t. $T^{-1}\mathcal{M}_0 \subset \mathcal{M}_0$ and*

$$\sum_{j\geq 0} |E(\varphi \mid T^{-j}\mathcal{M}_0)|_2 + \sum_{j\geq 0} |E(\varphi \mid T^{j}\mathcal{M}_0) - \varphi|_2 < \infty \qquad (*).$$

Then

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ T^i \stackrel{\text{distr}}{\longrightarrow} \mathcal{N}(0, \sigma)$$

where

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{i=0}^{n-1} \varphi \circ T^i \right)^2 d\nu.$$

Noninvertible case. *If T is not invertible, take $\mathcal{M}_0 = \mathcal{M}$ and disregard the second term in $(*)$. Otherwise same hypotheses and same conclusion.*

Remark. Suppose for instance that (T, ν) is a K-automorphism and $\mathcal{M}_0$ is such that $T^j\mathcal{M}_0 \uparrow \mathcal{M}$ and $T^{-j}\mathcal{M}_0 \downarrow$ the trivial σ-algebra as $j \to \infty$. Then $(*)$ could be viewed as an L^2-version of the K-property seen through the eyes of the random variable φ.

B. Condition for CLT and the Perron-Frobenius operator. Exponential decay of correlations (as defined in 1.4) alone does not imply $(*)$, but if $T : \Omega \circlearrowleft$ is noninvertible and admits some additional structures so that a Perron-Frobenius operator P can be defined, then $(*)$ can be expressed in terms of P. In particular, assuming that all relevant functions belong in a suitable space, then a gap in the spectrum of P implies that $\sum_{j\geq 0} |E(\varphi \mid T^{-j}\mathcal{M})|_2$ is a geometric series. Following [Ke] and putting what is there into a slightly broader context, we make this connection precise:

Let $T : (\Omega, \mathcal{M}, \nu) \circlearrowleft$ be noninvertible, and suppose that on $(\Omega, \mathcal{M})$ there is a reference measure m with respect to which T is nonsingular and $\nu = \rho m$ for some ρ with $\rho \geq c > 0$. Let $P(\varphi)$ be the density wrt m of the measure $T_*(\varphi m)$, and let $U : L^2(\nu) \circlearrowleft$ be the operator defined by $U(\varphi) = \varphi \circ T$. It is straightforward to verify that for all $\varphi \in L^2(\nu)$:

- $U^{*j}(\varphi) = P^j(\varphi\rho)/\rho$
- $U^j U^{*j}(\varphi) = E(\varphi \mid T^{-j}\mathcal{M})$ where U^* is the adjoint of U.

Using these, one sees immediately that

$$\begin{aligned} \int |E(\varphi \mid T^{-j}\mathcal{M})|^2 d\nu = \int |U^j U^{*j}\varphi|^2 d\nu &= \int |U^{*j}\varphi|^2 d\nu \\ = \int |(U^j U^{*j}\varphi) \cdot \overline{\varphi} d\nu &\leq |\varphi|_\infty \int |U^{*j}\varphi| \circ T^j d\nu \\ &= |\varphi|_\infty \int |P^j(\varphi\rho)| dm. \end{aligned}$$

C. Positivity of the variance. The following fact is communicated to me by Bill Parry (see also [PP]).

Lemma. *Let $T : (\Omega, \mathcal{M}, \nu) \circlearrowleft$ be an mpt and let $\varphi \in L^2(\nu)$ be s.t. $\int \varphi d\nu = 0$. Suppose also that $\int (\varphi \circ T^n)\varphi d\nu \to 0$ exponentially fast. Then $\frac{1}{n} \int \left(\sum_0^{n-1} \varphi \circ T^i \right)^2 d\nu \to 0$ iff $\varphi = \psi \circ T - \psi$ for some $\psi \in L^2(\nu)$.*

Note added in proof. Recent results [KV], [L2] (which have just come to the author's attention) have made it possible to relax the conditions in paragraphs A. and C. above, although the version reported here is entirely adequate for our purposes.

5.2. Proof of Theorem 3.

We now return to the setting and notations prior to 5.1.

Let $\varphi \in \mathcal{H}_\eta$. Again we lift to $\tilde{\varphi} : \Delta \to \mathbb{R}$ and note that a CLT for $\{\tilde{\varphi} \circ F^i\}$ implies one for $\{\varphi \circ f^i\}$. Also, once the convergence in distribution is proved, the positivity of σ^2 follows automatically from Theorem 2 and the lemma above. Observe, however, that technically we could not appeal to 5.1A, B and finish immediately, for our Perron-Frobenius operator is associated with $\overline{F} : \overline{\Delta} \circlearrowleft$ while $\tilde{\varphi}$ is defined on Δ. Here is one way to reconcile these differences:

Let $\overline{\mathcal{B}}$ be the Borel σ-algebra on $\overline{\Delta}$. Let $\mathcal{B}_0 := \{\overline{\pi}^{-1}\overline{A} : \overline{A} \in \overline{\mathcal{B}}\}$, and let $\overline{\varphi}_0 := E_{\tilde{\nu}}(\tilde{\varphi} \mid \mathcal{B}_0)$.

Claim #1: It suffices to show $\sum_{j \geq 0} |E_{\overline{\nu}}(\overline{\varphi}_0 \mid \overline{F}^{-j}\overline{\mathcal{B}})|_2 < \infty$ where $|\cdot|_2$ is wrt $\overline{\nu}$.

Claim #2: The sum above is finite.

To verify Claim #1, let $\hat{F} : (\hat{\Delta}, \hat{\nu}) \circlearrowleft$ denote the natural extension of $F : (\Delta, \tilde{\nu}) \circlearrowleft$. Then $\hat{\Delta}$ is homeomorphic to

$$\{t; \overline{x}_0, \overline{x}_1, \dots) \in (\Lambda/\Gamma^u) \times \overline{\Delta}^{\mathbb{N}} : \overline{F}\overline{x}_{i+1} = \overline{x}_i\}$$

where Λ/Γ^u is the quotient of Λ obtained by collapsing along γ^u-leaves. Let $\hat{\mathcal{B}}_0$ be the σ-algebra $\hat{\pi}^{-1}\mathcal{B}_0$ where $\hat{\pi} : \hat{\Delta} \to \Delta$ is the natural projection. Then $\hat{F}^{-1}\hat{\mathcal{B}}_0 \subset \hat{\mathcal{B}}_0$.

Let $\hat{\varphi} : \hat{\Delta} \to \mathbb{R}$ be the lift of $\tilde{\varphi}$. Noting that a CLT for $\{\hat{\varphi} \circ \hat{F}^i\}$ is equivalent to one for $\{\tilde{\varphi} \circ F^i\}$, we proceed to verify Gordin's condition $(*)$ for the conditional expectations $\hat{\varphi}_j := E_{\hat{\nu}}(\hat{\varphi} \mid \hat{F}^j\hat{\mathcal{B}}_0)$. For $j \geq 0$, $\hat{F}^j(\hat{\mathcal{B}}_0)$ is generated by sets of the form

$$\{(t; \overline{x}_0, \overline{x}_1, \dots) : t \in F^j\overline{\pi}^{-1}\{\overline{a}\},\ \overline{x}_0 = \overline{F}^j\overline{a}, \dots, \overline{x}_j = \overline{a}\}, \quad \overline{a} \in \overline{\Delta}.$$

Thus $|\hat{\varphi}_j - \hat{\varphi}| < C\alpha^{j\eta}$ and so $\sum_{j \geq 0} |\hat{\varphi}_j - \hat{\varphi}|_2 < \infty$. As for $\hat{\varphi}_{-j}, j \geq 0$, since the order of conditioning is immaterial, $\hat{\varphi}_{-j}$ can also be written as $E_{\overline{\nu}}(\overline{\varphi}_0 \mid \overline{F}^{-j}\overline{\mathcal{B}})$, and so it suffices to prove $\sum_{j \geq 0} |E_{\overline{\nu}}(\overline{\varphi}_0 \mid \overline{F}^{-j}\overline{\mathcal{B}})|_2 < \infty$ as claimed.

We are now in the situation of 5.1B. If we know $\overline{\varphi}_0\overline{\rho} \in X$, then, recalling that $\int \overline{\varphi}_0\overline{\rho} d\overline{m} = 0$, we could finish by saying

$$\int |P^j(\overline{\varphi}_0\overline{\rho})| d\overline{m} \le C_0'' \|P^j(\overline{\varphi}_0\overline{\rho})\| \le C\tau^j \|\overline{\varphi}_0\overline{\rho}\|.$$

Now $\overline{\varphi}_0\overline{\rho} \in X$ if $\overline{\varphi}_0 \in X$ (this uses the fact that $\overline{\rho} \mid \overline{\Delta}_\ell = \overline{\rho} \circ \overline{F}^{-\ell}$). To prove Claim #2, it remains only to show

Sublemma. *$\varphi \in \mathcal{H}_\eta \Rightarrow \overline{\varphi}_0 \in X$ provided that β in the definition of X satisfies $\beta \ge \alpha^{\min(\frac{1}{2},\eta)}$.*

Proof. This is a straightforward exercise in conditioning. In the spirit of the proof of Lemma 2, realize $\tilde{\nu}$ as a limit of $\tilde{\nu}_n = \frac{1}{n}\sum\limits_0^{n-1} F_*^i(m \mid \gamma_0 \cap \Lambda)$ where γ_0 is a fixed γ^u-leaf in Γ^u. (See also Section 3.) Recall that $\mathcal{M}_q(x) = \{y \in \Delta : F^i x, F^i y \in$ same $\Delta_{\ell,j}\ \forall i \le q\}$. Now pick x, x' lying the same $\Delta_{\ell,j}$. We assume x, x' are typical in the sense of the martingale convergence $E_{\tilde{\nu}}(\varphi \mid \mathcal{M}_q) \overset{\text{a.e.}}{\to} \overline{\varphi}_0$. Fix q very large. We will compare $E_{\tilde{\nu}_n}(\varphi \mid \mathcal{M}_q)$ at x and at x' for all large n.

Let $\{\omega_j\}$ be the collection of components of $\bigcup\limits_{i\le n}(F^i\gamma_0 \cap \mathcal{M}_q(x))$, and let $\{\omega_j'\}$ be the corresponding segments for x', ω_j and ω_j' taken to be on the same γ^u-leaf. We assume q is large enough that $\varphi \mid \omega_j \approx$ a constant which we denote by φ_j; similarly $\varphi \mid \omega_j' \approx \varphi_j' (=$ a constant). We assume also that the densities wrt m of $F_*^i(m \mid \gamma_0 \cap \Lambda)$ on ω_j and ω_j' are roughly constant, and call these constants respectively ρ_j and ρ_j'. Then

$$E_{\tilde{\nu}}(\varphi \mid \mathcal{M}_q)(x) \approx \frac{\sum_j \int_{\omega_j} \varphi_j \rho_j dm}{\sum_j \int_{\omega_j} \rho_j dm} = \frac{\sum_j \varphi_j\rho_j}{\sum_j \rho_j}$$

because $m(\omega_j)$ is independent of ω_j (Lemma 1(1)). There is of course a similar estimate for $E_{\tilde{\nu}}(\varphi \mid \mathcal{M}_q)(x')$. Now $|\varphi_j - \varphi_j'| \le C_\varphi \max\{d(\pi y, \pi y')^\eta : y \in \omega_j, y' \in \omega_j'\}$, which is $\lesssim C\alpha^{s(x,x')\eta}$; also $|\rho_j - \rho_j'| \lesssim C\rho_j \alpha^{\frac{1}{2}s(x,x')}$ (Lemma 2). An easy computation then gives

$$\left| \frac{\sum \varphi_j\rho_j}{\sum \rho_j} - \frac{\sum \varphi_j'\rho_j'}{\sum \rho_j'} \right| \le C\beta^{s(x,x')}$$

as claimed provided $\beta \ge \alpha^{\min(\frac{1}{2},\eta)}$. ∎

PART II. EXAMPLES OF SYSTEMS THAT FIT MODEL

In each of the settings in Sections 6-10, we will

- construct a set Λ with a hyperbolic product structure
- define a return map $f^R : \Lambda \circlearrowleft$
- verify properties (P1)-(P5) in Section 1
- estimate the measure of $\{R > n\}$.

If we succeed in all of this, and if the measure of $\{R > n\}$ decreases exponentially in n, then in our statements of results we will summarize by saying

"f fits the model of Part I with an exponential estimate for $\{R > n\}$",

and the conclusions of Theorems 2 and 3 will be abbreviated as

"(f, ν) has exponential decay of correlations and CLT".

We remark briefly on the construction of $f^R : \Lambda \circlearrowleft$. The choice of Λ is quite arbitrary; if (f, ν) is indeed mixing, it matters little where Λ is placed. If the existence of stable or unstable manifolds is a concern, take only points that approach the "bad set" sufficiently slowly. Technical considerations aside, f^R is defined by running f until some s-subset of Λ crosses over Λ completely in the u-direction; the part that has landed back in Λ is then deleted and we continue to iterate the rest – the key point here being that we do not view partial crossings of Λ as legitimate returns.

The reader will notice that while each of the settings below has its own technical aspects that require customized attention, the analyses needed to verify that they fit our model do not differ substantially. It is this recurrent pattern of proof that we hope can be mimicked in other situations.

6. Axiom A Attractors

6.1. Results and discussion.

Let $f : M \circlearrowleft$ be a C^2 diffeomorphism of a Riemannian manifold and let Σ be an attractor. By "attractor" we include also the Anosov case where $M = \Sigma$.

Theorem 4. (a) *f fits the model of Part I with an exponential estimate for $\{R > n\}$. As a consequence we obtain*

(b) *f admits an SRB measure ν on Σ; and*

(c) *if $f \mid \Sigma$ is topologically mixing, then (f, ν) has exponential decay of correlations and CLT.*

We remark that up to a finite cycle all Axiom A attractors are topologically mixing. The results in Theorem 4 are not new; they were first proved using Markov partitions (see [S1], [R1], [R2] and [B]). We would nevertheless like to illustrate our scheme of proof (which does not require *a priori* knowledge of Markov partitions) for this much studied class of diffeomorphisms. The proof of (a) is quite simple in dimension 2; the geometry of stable and unstable manifolds is more interesting in higher dimensions.

Remark. The picture could be quite different when uniform hyperbolicity is relaxed. In [HY] and [H], it is shown that for diffeomorphisms that are uniformly hyperbolic except for the presence of an indifferent fixed point, one could, by adjusting the derivatives at this point, arrange for the return time function R to have a variety of tail behaviors.

Expanding maps. A mapping $f : M \circlearrowleft$ of a compact Riemannian manifold is said to be uniformly expanding if $\exists \lambda > 1$ s.t. $|Df_x v| \geq \lambda |v|$ $\forall x \in M$ and all tangent vectors v. Our model in Part I needs to be modified for noninvertible maps, but the changes are quite obvious and are left to the reader.

Theorem 4'. *The results of Theorem 4 hold for C^2 uniformly expanding maps.*

The proofs are entirely parallel to – and considerably simpler than – those for Axiom A and will be omitted.

6.2. Construction of $f^R : \Lambda \circlearrowleft$.

Let $E^u \oplus E^s$ denote the usual splitting of the tangent bundle. For simplicity we assume the metric is "adapted", i.e. $\exists \alpha < 1$ s.t. $|Dfv| \leq \alpha|v|$ $\forall v \in E^u$ and $|Df^{-1}v| \leq \alpha|v|$ $\forall v \in E^s$. Let W^u denote unstable manifolds, $d^u(\cdot,\cdot)$ denote the distance measured along W^u, and $W^u_\delta(x) := \{y \in W^u(x) : d^u(x,y) \leq \delta\}$. We review a few properties of Axiom A attractors and set some notations:

(1) *Distortion along W^u_δ.* There exist $C, \delta_0 > 0$ s.t. $\forall x \in \Sigma$ and $\forall y \in W^u_{\delta_0}(x)$,

$$\log \frac{\det D(f^n|W^u)(f^{-n}x)}{\det D(f^n|W^u)(f^{-n}y)} \leq C d^u(x,y) \quad \forall n \geq 1.$$

(2) *Local product structure.* $\exists \delta_1 < \delta_0$ s.t. $\forall x \in \Sigma$, if $z_u \in W^u_{\delta_1}(x)$ and $z_s \in W^s_{\delta_1}(x) \cap \Sigma$, then $W^s_{\delta_0}(z_u)$ meets $W^u_{\delta_0}(z_s)$ transversally in exactly one point which we denote by $[z_u, z_s]$. For all $\delta_u, \delta_s \leq \delta_1$,

$$\mathcal{N}_{\delta_u,\delta_s}(x) := \{[z_u, z_s] : z_u \in W^u_{\delta_u}(x),\ z_s \in W^s_{\delta_s}(x)\}$$

is well defined and is an open neighborhood of $x \in \Sigma$. Clearly, $\mathcal{N}_{\delta_u,\delta_s}(x)$ has a hyperbolic product structure in the terminology of Section 1; it is the union of a disjoint collection of W^u-disks each one of which is homeomorphic to $W^u_{\delta_u}(x)$.

(3) *"u-crossings" and topological transitivity.* Let $A, B \subset \Sigma$ have hyperbolic product structure. We say that $f^n A$ "u-crosses" B if there is an s-subset A^s of A s.t. $f^n A^s$ is a u-subset of B. It follows from topological transitivity and standard hyperbolic arguments that if $A = \mathcal{N}_{\delta_u,\delta_s}(x)$ and $B = \mathcal{N}_{\delta'_u,\delta'_s}(x')$, then there exists N depending only on $\delta_u,\ \delta_s,\ \delta'_u,\ \delta'_s$ s.t. $f^n A$ u-crosses B for some $n \leq N$.

We now define $f^R : \Lambda \circlearrowleft$. Pick an arbitrary $\hat{x} \in \Sigma$, and let $\delta > 0$ be sufficiently small that $\mathcal{N}_{4\delta,\delta}(\hat{x})$ makes sense. Let us write $\mathcal{N}_c := \mathcal{N}_{c\delta,\delta}(\hat{x})$ for short, and let $\Lambda = \mathcal{N}_1$. Let $\Omega = W^u_\delta(\hat{x})$. To define f^R on Λ, it suffices to define it on Ω.

The idea of the construction is as follows. Let $\Omega_{n-1} := \Omega - \{R < n\}$, and suppose that we let those parts of Ω_{n-1} whose f^n-images u-cross Λ return at time n. Then $f^n\{R = n\}$ is the union of a finite number of topological disks (with jagged edges!) and $f^n\Omega_n$ is a large sheet with many "holes" of varying sizes corresponding to the different components of $\{R \le n\}$. For each ε-disk in $f^n\Omega_n$, property (3) above guarantees that a fixed percentage will return within a fixed number of iterates. This is what is going to give the exponential tail estimate for R. Points that are not contained in full ε-disks in $f^n\Omega_n$ form "collars" around the holes. We will argue that they escape from these collars exponentially fast because $f|W^u$ is uniformly expanding.

In the formal argument it seems simpler to fix all the expiration times for the entire "collar" at the moment a "hole" is created. For bookkeeping purposes we introduce a partition on $\mathcal{N}_2 - \mathcal{N}_1$. Let

$$\hat{I}_k := \{x \in W^u_{2\delta}(\hat{x}) : \delta(1 + \alpha^k) < d^u(x, \hat{x}) \le \delta(1 + \alpha^{k-1})\},$$

so that $\hat{I}_1$ is the outermost ring in $W^u_{2\delta}(\hat{x})$ and $\bigcap_{k \ge 1} \hat{I}_k = \Omega$. Let $I_k := \hat{\pi}^{-1}\hat{I}_k$ where $\hat{\pi} : \mathcal{N}_2 \to W^u_{2\delta}(\hat{x})$ is projection by sliding along W^s-leaves. Let R_0 be the first time when returns are allowed (see (P2)); let $\Omega_{R_0} := \Omega$ and set $t_{R_0} \equiv 0$. For $n > R_0$, the mechanics of the induction is as follows. We assume at the beginning of step n that we are handed a set $\Omega_{n-1} := \Omega - \{R < n\}$ and a function t_{n-1} defined on Ω_{n-1} ($t_{n-1}(x) = k$ means that $f^{n-1}x$ will stay in a "collar" for the next k iterates). Let $A_{n-1} = \{t_{n-1} = 0\}$, $B_{n-1} = \{t_{n-1} > 0\}$. Let $\varepsilon > 0$ be a small number to be specified, and let $A^{(\varepsilon)}_{n-1} = \{x \in \Omega_{n-1} : d^u(f^n x,\ f^n A_{n-1}) < \varepsilon\}$. To define $\{R = n\}$, we let $\{D^4_j\}_{j=1,2,\dots}$ be those components of $A^{(\varepsilon)}_{n-1} \cap f^{-n}\mathcal{N}_4$ with the property that the s-subsets of Λ through them are mapped under f^n onto u-subsets of $\mathcal{N}_4$. Let $D^i_j = D^4_j \cap f^{-n}\mathcal{N}_i$ for $i = 1, 2, 3$. We declare that $R|D^1_j = n$ and define t_n on $\Omega_n := \Omega_{n-1} - \bigcup_j D^1_j$ as follows: for $x \in \bigcup_j (D^2_j - D^1_j)$, let $t_n(x) = k$ if $f^n x \in I_k$; set $t_n = 0$ elsewhere on $A^{(\varepsilon)}_{n-1}$; and for all other x, reset $t_n(x) = t_{n-1}(x) - 1$.

In order for these definitions to make sense, we must have $t_{n-1}|A^{(\varepsilon)}_{n-1} \le 1$, otherwise some points very near the boundary of Ω_{n-1} may suddenly be assigned $t_n = 0$. This is tantamount to requiring that the "collars" (i.e. $f^n\{t_n > 0\}$) around different holes be pairwise disjoint. We claim that this is true if ε is sufficiently small. To see this, let Q be a component of $\{R = n - i\}$ for some $i > 0$, and let Q_k be that part of the collar around Q that u-crosses I_k under f^{n-i}. We assume the desired picture has been valid up to this point, so that $t_{n-1}|Q_i = 1$ and $t_{n-1} = 0$ in a neighborhood of $Q \cup (\bigcup_{k \ge i} Q_k)$. Let $\partial_1 Q_i$ and $\partial_2 Q_i$ be the two components of ∂Q_i. We estimate the minimum d^u-distance between $f^n(\partial_1 Q_i)$ and $f^n(\partial_2 Q_i)$. Let

$w_1 \in f^{n-i}(\partial_1 Q_i),\ w_2 \in f^{n-i}(\partial_2 Q_i)$. Then

$$\begin{aligned} d^u(\hat{\pi} w_1, \hat{\pi} w_2) &\le \alpha^i d^u(f^i \hat{\pi} w_1, f^i \hat{\pi} w_2) \\ &\le c\alpha^i \{d(f^i \hat{\pi} w_1, f^i w_1) + d^u(f^i w_1, f^i w_2) + d(f^i w_2, f^i \hat{\pi} w_2)\} \\ &\le c\alpha^i \{d^u(f^i w_1, f^i w_2) + 2\delta\alpha^i\}; \end{aligned}$$

but we also know that

$$d^u(\hat{\pi} w_1, \hat{\pi} w_2) \approx \delta\alpha^i(1-\alpha).$$

Comparing these two estimates, we see that $d^u(f^i w_1, f^i w_2) \ge$ some $\varepsilon_0 > 0\ \forall i \ge$ some i_0; hence it is $\ge$ some $\varepsilon_1\ \forall i$. Choose $\varepsilon < \varepsilon_1$.

We assume also $\varepsilon < \delta$, so that $f^n x \notin \mathcal{N}_3\ for\ x \in D_j^4 \cap (A_{n-1}^{(\varepsilon)} - A_{n-1}),\ i.e.\ (D_j^4 \cap A_{n-1}) \supset D_j^3$.

We have defined the return map f^R on a subset of Λ that we will show in 6.3 includes a set of full measure on every unstable leaf. Modulo this fact, (P1) and (P2) have been arranged. Here it is natural to define $s_0(x,y)$ be the largest n with $d(f^i x, f^i y) < \delta_0\ \forall i \le n$. (Formally this does not satisfy all the requirements of $s_0(\cdot,\cdot)$ in 1.2, but as we have explained in 1.3, the precise definition of s_0 is not important.) (P3)-(P5) are standard for uniformly hyperbolic systems.

6.3. Estimation of $\mu_\Omega\{R > n\}$.

Recall that $\Omega_{n-1} = \Omega - \{R < n\}$ is the disjoint union $A_{n-1} \cup B_{n-1}$, where B_{n-1} is a neighborhood of $\partial\Omega_{n-1}$, and $A_{n-1}^{(\varepsilon)}$ is a small neighborhood of A_{n-1}. As we go from step $n-1$ to step n, points near the outer edges of the collars move from B_{n-1} to A_n, some points in A_{n-1} return to Λ, and some points go from A_{n-1} to B_n as new collars are formed. The following two sublemmas describe the rules that govern this 3-way exchange of mass. Recall that μ_Ω denotes the Riemannian measure on Ω.

Sublemma 1. (Flow of mass from A_{n-1})
(i) $\exists\ a, c_1\ with\ a + c_1 < 1$ s.t. $\forall n,$
  * $\mu_\Omega(A_{n-1} \cap B_n) \le a\mu_\Omega(A_{n-1})$
  * $\mu_\Omega(A_{n-1} \cap \{R = n\}) \le c_1\mu_\Omega(A_{n-1})$.
    (This inequality is used only to simplify the argument.)

(ii) $\exists\ c_2 > 0\ and\ N = N(\varepsilon)$ s.t. $\forall n,$
  * $\mu_\Omega\left(\bigcup\limits_{i=0}^{N}\{R = n+i\}\right) \ge c_2\mu_\Omega(A_{n-1}^{(\varepsilon)}) \ge c_2\mu_\Omega(A_{n-1})$

Sublemma 2. (Flow of mass from B_{n-1}) $\exists b > 0$ s.t. $\forall n,$
  * $\mu_\Omega(B_{n-1} \cap A_n) \ge b\mu_\Omega(B_{n-1})$.

Let us set $R = \infty$ on the part of Ω where it is not defined. To prove that $\mu_\Omega\{R > n\} \le C_0\theta_0^n$, it suffices to prove that $\mu_\Omega\left(\bigcup\limits_{i=0}^{N}\{R = n+i\}\right) \ge c_2'\mu_\Omega(\Omega_{n-1})$ for some $c_2' > 0$. This will follow from Sublemma 1(ii) once we know that $(\mu_\Omega(A_n)/\mu_\Omega(B_n))$ is bounded away from 0 for all n. To see this, observe that whenever $(a+c_1)\mu_\Omega(A_{n-1}) < b\mu_\Omega(B_{n-1})$, one step later we will have $\mu_\Omega(A_n) > \mu_\Omega(A_{n-1})$ and $\mu_\Omega(B_n) < \mu_\Omega(B_{n-1})$, i.e. the situation improves. It remains to prove the two sublemmas.

Proof of Sublemma 1(i). We will show $\mu_\Omega(A_{n-1} \cap \{t_n > 0\}) \leq a\mu_\Omega(A_{n-1})$ by working with the neighborhood of one component of $\{R = n\}$ at a time. Using the notations in 6.2, letting $\hat\Omega = W^u_{4\delta}(\hat x)$, and remembering that $(D^4_j \cap A_{n-1}) \supset D^3_j$, we have

$$\frac{\mu_{\hat\Omega}(D^2_j - D^1_j)}{\mu_{\hat\Omega}(D^3_j)} \approx \frac{\mu_{f^n\hat\Omega}(f^nD^2_j - f^nD^1_j)}{\mu_{f^n\hat\Omega}(f^nD^3_j)} \approx \frac{\mu_{\hat\Omega}(W^u_{2\delta}(\hat x) - W^u_\delta(\hat x))}{\mu_{\hat\Omega}(W^u_{3\delta}(\hat x))} < 1.$$

More precisely, the last fraction is clearly < 1, and, using the fact that the Radon-Nikodym derivative of $\hat\pi : f^nD^4_j \to \hat\Omega$ is bounded above and below, the second fraction is also uniformly bounded away from 1. Finally, the first and second fractions are comparable via the distortion estimate in 6.2.

The second inequality is proved similarly. ∎

Proof of Sublemma 2. We observe that for all i,

$$\frac{\mu_{\hat\Omega}\left(\cup_{k=i+1}^\infty \hat I_k\right)}{\mu_{\hat\Omega}\left(\cup_{k=i}^\infty \hat I_k\right)} \approx \frac{[\delta(1+\alpha^{k+1})]^d - \delta^d}{[\delta(1+\alpha^k)]^d - \delta^d} \approx \alpha$$

where $d = \dim W^u$. This together with the same Radon-Nikodym derivative and distortion estimates as above give

$$\mu_\Omega\{t_{n-1} = 1\} \geq b\mu_\Omega\{t_{n-1} > 0\} = b\mu_\Omega(B_{n-1})$$

for some b. ∎

Proof of Sublemma 1(ii). Let $\mathcal{E}_n = \{z_j\}$ be a maximal set in f^nA_{n-1} with the property that $B^u(z_j, 5\delta)$ are pairwise disjoint (B^u = balls in the d^u-metric). We observe that

(i) $\bigcup_{z_j \in \mathcal{E}_n} B^u(z_j, 11\delta) \supset f^nA^{(\varepsilon)}_{n-1}$,

(ii) $\exists\, N(\varepsilon)$ *s.t.* for each j, $\exists\, i \leq N(\varepsilon)$ *s.t.* $f^iB^u(z_j, \varepsilon)$ $u-$crosses $\mathcal{N}_4$.

(i) is true because $\bigcup_{z_j \in \mathcal{E}_n} B^u(z_j, 10\delta)$ clearly $\supset f^nA_{n-1}$, and $\varepsilon < \delta$. (ii) is a consequence of property (3) in 6.2. Let $n_j \geq 0$ be the first time when $t_{n+n_j}|f^{-n}B^u(z_j,\varepsilon)$ is not identically zero. Then $f^{n_j}B^u(z_j,\varepsilon)$ must meet $\mathcal{N}_2$ and $f^{-n}B^u(z_j, \varepsilon + 4\delta)$ must contain a component of $\{R = n + n_j\}$. We know from (ii) that $n_j \leq N(\varepsilon)$.

Now (i) together with the standard distortion estimate says that

$$\mu_\Omega(A^{(\varepsilon)}_{n-1}) \leq \sum_j \mu_\Omega f^{-n}B^u(z_j, 11\delta) \leq \text{ const } \sum_j \mu_\Omega f^{-n}B^u(z_j, 5\delta);$$

and the disjointness of the $B^u(z_j, 5\delta)$'s together with the observation in the last paragraph guarantee that

$$\sum_j \mu_\Omega f^{-n} B^u(z_j, 5\delta) \leq \text{ const} \cdot \mu_\Omega\left(\cup_{i\leq N}\{R = n+i\}\right).$$

$\blacksquare$

7. Piecewise Hyperbolic Maps

7.1. Results and discussion.

We consider in this section piecewise uniformly hyperbolic diffeomorphisms in 2-dimensions. More precisely, we consider $f : M \circlearrowleft$ where M is a compact 2-dimensional Riemannian manifold possibly with boundary. We assume:

(H1) *f is a piecewise C^2 diffeomorphism, i.e. there is a finite number of pairwise disjoint open regions $\{M_i\}$ whose boundaries are C^1 curves of finite length such that $\cup \bar{M}_i = M$, $f|(\cup M_i)$ is $1-1$ and f restricted to each $\bar{M}_i$ is a C^2 diffeomorphism onto its image.*

We will sometimes refer to $S := M - \cup M_i$ as the "singularity set". Note that we allow $f(M) \subsetneqq M$; in particular, M could be a trapping region for an attractor.

(H2) *f is uniformly hyperbolic, i.e. there exist two continuous Df-invariant families of cones C^u and C^s defined on all of M and a number $\lambda > 1$* s.t.

$$|Dfv| \geq \lambda |v| \quad \forall v \in C^u,$$
$$|Dfv| \leq \lambda^{-1}|v| \quad \forall v \in C^s.$$

(H3) *On $f(M)$, tangent vectors to S are bounded away from C^u.*

We call γ a *u-curve* if all of its tangent vectors are in C^u; *s-curves* are defined similarly.

(H4) $\exists\ \bar{\varepsilon}, \bar{\delta} > 0$ *and $N, K \in \mathbb{Z}^+$ with $K < \lambda^N$ s.t. the following holds: if γ is a u-curve with $\ell(\gamma) \leq \bar{\varepsilon}$, and $0 \leq \delta_i \leq \bar{\delta}$, $i = 0, 1, ...N-1$, then the set $\{x \in \gamma : d(f^i x, S) > \delta_i$ for $i = 0, \ldots, N-1\}$ has $\leq K$ connected components.*

The motivation for (H4) will be given shortly.

Theorem 5. *Let $f : M \circlearrowleft$ satisfy* (H1)-(H4). *Then*

(a) *f fits the model of Part I with an exponential estimate for $\{R > n\}$; as a consequence we obtain*

(b) *f admits an SRB measure ν; and*

(c) *if (f^n, ν) is ergodic $\forall n \geq 1$, then (f, ν) has exponential decay of correlations and CLT.*

The existence of SRB measures for Lozi or Lozi-like mappings is first proved independently in [CL], [Ry] and [Y1], and extended slightly in [P2]. Subexponential decay (or "stretched" exponential decay) of correlations and CLT is proved in e.g. [C2]. The area preserving case of (c) is first proved in [L1].

The following are perhaps the two biggest differences between Axiom A and the present setting:

(1) *Arbitrarily short W^u- and W^s-curves and the absence of a local product structure.* This means that topologically Λ is necessarily the product of two Cantor sets, and it is not likely to be open in M or in the attractor. An immediate problem is how to make the Cantor sets "match" when the box spanned by Λ u-crosses itself.

(2) *Growth of u-curves.* Consider the following scenario. Suppose that $|Dfv| \approx \frac{3}{2}|v|\ \forall v \in C^u$, and the u-curve γ gets folded roughly around the middle when f is applied. Assume further that each component of $f^i\gamma$ gets folded in the same way for $i = 1, 2, \dots$. Then $f^i\gamma$ behaves increasingly like a point even though its total length is growing exponentially! We do not know if this phenomenon can actually occur *ad infinitum*, but to have uniform estimates we need an assumption of the following type:

(H4′) $\exists\ N, K \in \mathbb{Z}^+$ with $K < \lambda^N$ s.t. if γ is a sufficiently short u-curve, then $f^N\gamma$ has $\leq K$ smooth components.

For technical reasons it is convenient to assume a little more, hence (H4).

Piecewise expanding maps of $[0,1]$. Here S consists of a finite number of points, (H4) is automatic, and a much simplified version of our proof gives

Theorem 5'. *The results of Theorem 5 hold for piecewise C^2 expanding maps of the interval.*

The existence of absolutely continuous invariant measures for these maps is first proved in [LaY]; (c) is first proved in [HK].

7.2. Preliminaries: stable and unstable manifolds.

By a *curve* we always mean a connected smooth curve. If γ is a curve, the connected components of $f^n\left(\gamma - \bigcup_{i=0}^{n-1} f^{-i}S\right)$ are called the *components* of $f^n\gamma$. We call γ a local stable manifold only if $f^n\gamma$ has exactly one component for all $n \geq 0$. An analogous remark with $n \leq 0$ applies to local unstable manifolds. The local stable manifold of length ε through x, written $W^s_\varepsilon(x)$, is defined to be $\{y \in W^s_{loc}(x) : d^s(x,y) \leq \varepsilon\}$. The following are standard:

(a) If $x \in M$ satisfies $d(f^n x, fS) > \lambda^{-n}\varepsilon\ \forall n > 0$, then $W^s_\varepsilon(x)$ exists. (See e.g. [KS].) In an analogous manner, if $f^{-n}x \in M\ \forall n \geq 0$ and $d(f^{-n}x, S) > \lambda^{-n}\varepsilon\ \forall n < 0$, then $W^u_\varepsilon(x)$ exists.

(b) $\exists C > C' > 0$ s.t. if γ is a u-curve with curvature $\leq C'$, then all components of $f^n\gamma$, all $n \geq 0$, have curvatures $\leq C$; moreover the distortion estimate in 6.2 holds on each component of $f^n\gamma$. The proofs are identical to those for Axiom A.

We use the criterion above to establish the existence of (many) stable and unstable manifolds. In the conservative case, if μ is Lebesgue measure and $U_\varepsilon(S)$ denotes the ε-neighborhood of S, then $\exists C > 0$ s.t. $\mu U_\varepsilon(S) < C\varepsilon \ \forall \varepsilon > 0$. Since

$$\sum_{n=0}^{\infty} \mu(f^n U_{\lambda^{-n}\varepsilon}(S)) = \sum_{n=0}^{\infty} \mu(U_{\lambda^{-n}\varepsilon}(S)) < \infty,$$

we have, by the Borel-Cantelli Lemma, the μ-a.e. existence of $W^u_{\varepsilon(x)}(x)$ for some measurable function $\varepsilon > 0$. The same argument gives an abundance of stable and unstable manifolds in the dissipative case once we prove

Sublemma 1. *Every f that satisfies* (H1)-(H4) *has a (nonatomic) invariant Borel probability measure $\hat{\mu}$ with the property that for some $C > 0$, $\hat{\mu} U_\varepsilon(S) < C\varepsilon \ \forall \varepsilon > 0$.*

We postpone the proof of Sublemma 1 to 7.4 but will use the existence of stable and unstable manifolds freely from this point forward.

7.3. Construction of $f^R : \cup \Lambda^{(i)} \circlearrowleft$.

Without specific knowledge of f and without transitivity assumptions, it is hard to know *a priori* where to place Λ. One solution is to first deploy copies of Λ everywhere; call them $\Lambda^{(1)}, \ldots, \Lambda^{(k)}$. The $\Lambda^{(i)}$'s may overlap, and they do not necessarily cover M, but they will be chosen to capture enough of the dynamics for our purposes – and it will become clear as the proof progresses that one of these $\Lambda^{(i)}$'s would have sufficed.

Let δ_0 be s.t. (i) $A_{\delta_0} := \{x \in M : W^u_{3\delta_0}(x) \text{ exists}\} \neq \phi$ and (ii) $20\delta_0 \leq \bar{\varepsilon}$ where $\bar{\varepsilon}$ is as in (H4). Let $\delta_1 \ll \delta_0$ be a small positive number to be determined. For $x \in A_{\delta_0}$, we define

$$\Omega(x) := W^u_{\delta_0}(x)$$

and

$$\Omega_\infty(x) := \{y \in \Omega(x) : d(f^n y, S) > \delta_1 \lambda^{-n} \ \forall n \geq 0\}.$$

Sublemma 2. $\exists c > 0$ s.t. *if δ_1 is sufficiently small, then $\mu_{\Omega(x)}(\Omega_\infty(x)) > c \ \forall x \in A_{\delta_0}$.*

Sublemma 2 is proved in 7.4. We fix a small enough δ_1 for the rest of Section 7. For future convenience we make the following small alteration in the above definition of $\Omega_\infty(x)$. Suppose inductively that Ω_{n-1} is defined. Let $\{\omega_i\}$ be the connected components of $\{y \in \Omega_{n-1} : d(f^n y, S) < \delta_1 \lambda^{-n}\}$. Delete ω_i from Ω_{n-1} if and only if the minimum distance between $f^n \omega_i$ and S is $< \frac{1}{2}\delta_1 \lambda^{-n}$. (This guarantees that no arbitrarily small gaps in Ω_{n-1} are created.) Let Ω_n be the resulting set, and define $\Omega_\infty = \bigcap_n \Omega_n$. Thus if ω is a component of Ω_n, then $f^n\omega$ is a connected smooth curve with $d(f^n\omega, S) \geq \frac{1}{2}\delta_1 \lambda^{-n}$; in particular, $f^{n+1}\omega$ is also a connected smooth curve.

Let δ be such that $10\delta < \delta_1$. Let us verify that for all $x \in A_{\delta_0}, W^s_{5\delta}(y)$ exists $\forall y \in \Omega_\infty(x)$. To see this, let γ be an s-curve of length $10\delta\lambda^{-n}$ centered at $f^n y$, and let γ_0 be the component of $f^{-n}\gamma$ containing y. We claim that γ_0 has length $\geq 10\delta$. The only way it could be shorter is if it is cut when brought back by f^{-n} i.e. there exists $y' \in \gamma_0$ with $d(y, y') \leq 5\delta$ s.t. $f^j y' \in fS$ for some $0 < j \leq n$, or equivalently, $f^j y' \in S$ for some $0 \leq j < n$. That is impossible, for $d(f^j y', S) \geq d(f^j y, S) - d(f^j y,\ f^j y')$, which must be > 0 since the first term is $\geq \frac{1}{2}\delta_1\lambda^{-j} > 5\delta\lambda^{-j}$ and the second is $\leq 5\delta\lambda^{-j}$. It is well known that $W^s_{5\delta}(y)$ is a limit of curves of the type γ_0 with $n \to \infty$.

Associated with each $x \in A_{\delta_0}$ we construct a set $\Lambda(x)$ with a hyperbolic product structure: let $\Gamma^s(x) = \{W^s_\delta(y) : y \in \Omega_\infty(x)\}$, and let $\Gamma^u(x) = \{$all W^u_{loc}-curves that meet every $\gamma^s \in \Gamma^s(x)$ and which extend $> \delta_0$ on both sides beyond the curves in $\Gamma^s(x)\}$. These two families define $\Lambda(x)$.

We now choose the $\Lambda^{(i)}$'s mentioned at the beginning of this subsection. Let $Q(x)$ be a rectangular shaped region with the following properties: $Q(x) \supset \Lambda(x)$; it contains x in its interior, and $\partial Q(x)$ is made up of 2 u-curves and 2 s-curves. The 2 u-curves are roughly $2\delta_0$ in length; they are either from $\Gamma^u(x)$ or they do not meet any element of $\Gamma^u(x)$. The 2 s-curves are approximately 2δ long and have the same property with respect to $\Gamma^s(x)$. Let $\hat{Q}(x)$ be a proper *u-subrectangle* of $Q(x)$, i.e. it shares the s-boundaries of $Q(x)$, and its u-boundaries, which must have the same properties as those of $Q(x)$, are strictly inside $Q(x)$. Let $int(\cdot)$ denote the interior of a set, and view $\{int(\hat{Q}(x)) : x \in A_{\delta_0}\}$ as an open cover of A_{δ_0}. Since A_{δ_0} is clearly compact, one may choose a finite subcover $\{int\hat{Q}(x_1), \ldots, int\hat{Q}(x_k)\}$. Let $\Lambda^{(i)} := \Lambda(x_i)$.

We record below two very important facts:

(1) Let ω be a connected component of Ω_n and let Q_ω be an s-subrectangle of $Q(x)$ corresponding to ω (there is some slight ambiguity in the definition of Q_ω but it will be clear what we mean). We claim that $f^j Q_\omega \cap S = \phi\ \forall j \leq n$. In fact, if by an *$n$-stable curve* we mean a curve whose n^{th} iterate has exactly one component and whose tangent vectors remain in the stable cones C^s through the n^{th} iterate, then we claim that Q_ω can be foliated with $(n+1)$-stable curves interpolating between the elements of Γ^s. To see this, observe that by definition, $d(f^n y, S) > \frac{1}{2}\delta_1\lambda^{-n}\ \forall y \in \omega$, so it is entirely possible to pass a continuous family $\{\mathcal{F}(\cdot)\}$ of 1-stable curves through each point of $f^n\omega$ extending $\geq \delta\lambda^{-n}$ on each side. We have argued that the pullback of $\{\mathcal{F}(\cdot)\}$ by f^{-n} foliates Q_ω.

(2) Every $W^u_{10\delta_0}$-curve γ u-crosses one of the $\hat{Q}(x_i)$'s with two segments of length $\geq 2\delta_0$ sticking out at each end. This is true because the mid-point of γ belongs in A_{δ_0} and so must lie in some $\hat{Q}(x_i)$.

Next we define the return map $f^R : \underset{i}{\cup}\ \Lambda^{(i)} \circlearrowleft$. Let i be fixed throughout, and let $\Omega_n = \Omega_n(x_i)$. Let us first define R on Ω_∞. We let $\tilde{\Omega}_n = \Omega_n - \{R \leq n\}$ and introduce a partition $\tilde{\mathcal{P}}_n$ on $\tilde{\Omega}_n$ as follows. Let $R_1 \geq R_0$ (where R_0 is as in (P2)) be s.t. $\forall n \geq R_1$, if ω is a component of Ω_n s.t. $f^n\omega$ u-crosses $\hat{Q}(x_j)$ for some j, then the entire rectangle $f^n Q_\omega$ u-crosses $Q(x_j)$. The definitions of $\tilde{\mathcal{P}}_n$ are different before and after time R_1.

For $n < R_1$: let $\tilde\omega \in \tilde{\mathcal{P}}_{n-1}$, and let ω be a component of $\tilde\omega \cap \Omega_n$. Then $\omega \subset \tilde\Omega_n$, and $\tilde{\mathcal{P}}_n \mid \omega$ is defined as follows:

* if $\ell(f^n\omega) \le 10\delta_0$, then $\omega \in \tilde{\mathcal{P}}_n$;
* if $\ell(f^n\omega) > 10\delta_0$, then we let $\tilde{\mathcal{P}}_n|\omega$ be a partition of ω into segments $\omega_1 \cup \cdots \cup \omega_m$ with $10\delta_0 \le \ell(\omega_k) < 20\delta_0$ for each k.

For $n \ge R_1$: let ω and $\tilde\omega$ be as above, and put $\omega \in \tilde{\mathcal{P}}_n$ as before if $\ell(f^n\omega) \le 10\delta_0$. If $\ell(f^n\omega) > 10\delta_0$, we partition ω as above and select for each ω_k a $\Lambda^{(j)}$, $j = j(\omega_k)$, s.t. $f^n\omega_k$ u-crosses $\hat Q(x_j)$ with segments $\ge 2\delta_0$ in length sticking out on each side. (See Fact (2) above.) On $\omega_k \cap f^{-n}\Lambda^{(j)}$, we declare that $R = n$. We put $\omega_k - \{R = n\}$ in $\tilde\Omega_n$ and let $\tilde{\mathcal{P}}_n$ be the partition which divides $\omega_k - \{R = n\}$ into its (infinitely many) connected components.

It is now time to confront the problem of "matching of Cantor sets" alluded to in 7.1.

Sublemma 3. *Let $n \ge R_1$, and let ω be a segment contained in $\Omega_n = \Omega_n(x_i)$ whose f^n-image u-crosses some $\hat Q(x_j)$ with $\ge 2\delta_0$ sticking out on each side (e.g. ω is one of the ω_k's in the $n \ge R_1$ case above). Then $\omega \cap \Omega_\infty \cap f^{-n}\Lambda^{(j)} \ne \phi$; and if A is the smallest s-subset of $\Lambda^{(i)}$ containing $\omega \cap \Omega_\infty \cap f^{-n}\Lambda^{(j)}$, then $f^n(A)$ is a u-subset of $\Lambda^{(j)}$.*

Proof. We will prove $f^n(\omega\cap\Omega_\infty) \supset f^n\omega\cap\Lambda^{(j)}$, which would imply the first assertion. Consider $y \in f^n\omega\cap\Lambda^{(j)}$, and let $x \in \omega$ be s.t. $f^n x = y$. We must show that $x \in \Omega_\infty$. Since $\omega \subset \Omega_n$, $x \in \Omega_n$ by definition. To see that $d(f^m x, S) > \delta_1\lambda^{-m}$ $\forall m > n$, let y' be the point in $\Omega_\infty(x_j)$ with $y \in W^s_\delta(y')$, and observe that $d(f^m x, S) = d(f^{m-n}y, S) \ge d(f^{m-n}y', S) - d(f^{m-n}y', f^{m-n}y) > \frac{4}{10}\delta_1\lambda^{-(m-n)}$ which we may assume is $\ge \delta_1\lambda^{-m}$ for $n \ge R_1$.

To prove $f^n A \subset \Lambda^{(j)}$, let $x \in A$, and let x' be the point in ω s.t. $x \in \gamma^s(x')$. Then $f^n x' \in \Lambda^{(j)}$ by choice of A, and $f^n x \in \gamma^s(f^n x')$. Also, it follows from Fact (1) above that $f^n(\gamma^u(x') \cap Q_\omega)$ lies in one component and sticks out by $> 2\delta_0$ on each side of $Q(x_j)$. Hence it must be one of the curves in $\Gamma^u(x_j)$. The assertion that $f^n A$ is a u-subset of $\Lambda^{(j)}$ follows from the product structure of $f^n A$ and the fact that $f^n(\omega \cap \Omega_\infty) \supset f^n\omega \cap \Lambda^{(j)}$. ■

We remark that Sublemma 3 does not assert that $f^n(\omega \cap \Omega_\infty) \subset \Lambda^{(j)}$; indeed much of 7.5 is concerned with the many small bits of $f^n(\omega \cap \Omega_\infty)$ that fall through the gaps of $\Lambda^{(j)}$.

We discuss the status of (P1)-(P5). Instead of one Λ, we have constructed a finite number of sets with hyperbolic product structures, namely $\Lambda^{(1)}, \ldots, \Lambda^{(k)}$. The positivity of Lebesgue measure of $\Lambda^{(i)} \cap \gamma^u$ follows from Sublemma 2 (applied to Ω) and from the absolute continuity of Γ^s (i.e. (P5)). This completes (P1). Sublemma 3 defines f^R on a subset of $\cup\Lambda^{(i)}$, but it remains to show that $R < \infty$

on a set of full measure on each $\gamma^u \cap \Lambda^{(i)}$, or equivalently, on $\Omega \cap \Lambda^{(i)}$. A natural separation time for $x, y \in \gamma^u$ is the largest n s.t. (i) $f^n x$ and $f^n y$ lie in the same component of $f^n \gamma^u$ and (ii) they are $\leq 30\delta_0$ apart. (For a formal definition of $s_0(\cdot, \cdot)$ with the properties in 1.2, one should require in (ii) that the smallest s-subrectangle of Λ containing x and y have diameter $\leq 30\delta_0$.) Let us give an explicit construction of $\{\Delta_{\ell,j}\}$ in terms of the $\tilde{\mathcal{P}}_\ell$'s. To do this it suffices to specify for each ℓ a finite partition $\mathcal{P}_\ell$ on $\tilde{\Omega}_\ell$ for which $\tilde{\mathcal{P}}_\ell$ is a refinement: let each element of $\mathcal{P}_\ell$ be of the form $\gamma \cap \tilde{\Omega}_\ell$ where γ is a subsegment of Ω with the property that the points in γ are still "together" ℓ steps later. (P3)-(P5) are virtually indistinguishable from those of Axiom A. In summary, it remains only to supply the proofs of Sublemmas 1 and 2 and to prove $R < \infty$ a.e. on $\Omega \cap \Lambda^{(i)}$.

7.4. A growth lemma.

In this subsection we introduce some stopping times ideas and prove a growth lemma for u-curves. We will carry this out in the setting of Sublemma 2; a similar argument is used in the proof of Sublemma 1 (which is given after that of Sublemma 2). Let us assume for simplicity that the N in (H4) is equal to 1.

Proof of Sublemma 2. Let Ω and Ω_n be as defined at the beginning of 7.3. We introduce a sequence of stopping times $T_1 < T_2 < \cdots$ on subsets of Ω as follows. For $x \in \Omega$, let $T_1(x)$ be the smallest $n > 0$ s.t. the component of $f^n\Omega_{n-1}$ containing $f^n x$ has length $> \bar{\varepsilon}$ (where $\bar{\varepsilon}$ is as in (H4)). If no such n exists, or if x is eliminated from Ω_n before T_1 is reached, we will say $T_1(x)$ is not defined. Let $\Theta_1 := \{x \in \Omega : T_1(x)$ is defined$\}$. Then Θ_1 is the disjoint union of a countable number of segments $\{\omega\}$ each one of which belongs in some Ω_{j-1} and satisfies $T_1|\omega = j$. On Θ_1, define $T_2(x) =$ the smallest $n > T_1(x)$ s.t. the component of $f^n\Omega_{n-1}$ containing $f^n x$ has length $> \bar{\varepsilon}$. Let $\Theta_2 := \{x \in \Theta_1 : T_2(x)$ is defined$\}$, and so on.

We begin by estimating the total measure on Ω deleted strictly before T_1. Since $\ell(\Omega) \leq \bar{\varepsilon}$, (H4) says that Ω_0, which is $\approx \{x \in \Omega : d(x, S) > \delta_1\}$, has $\leq K$ connected components. We stop considering those on which $T_1 = 1$. By (H4), each of the rest intersects Ω_1 in $\leq K$ components. We see inductively that $\Omega_{n-1} - \{T_1 < n\}$ has $\leq K^n$ components. Consider one of these, ω, with $T_1|\omega \neq n$. Then $f^n\omega$ has length $\leq \bar{\varepsilon}$, and it intersects $U_{\delta_1\lambda^{-n}}(S)$ in $\leq K+1$ segments (actually it is not exactly $U_{\delta_1\lambda^{-n}}$ that we want; see 7.3). Since all u-curves intersect S transversally with uniform bounds, each subsegment of $f^n\omega$ removed has length $\leq C\delta_1\lambda^{-n}$. Pulling back to ω and summing first over all ω in $\Omega_{n-1} - \{T_1 \leq n\}$ and then over n, we obtain that

$$\text{the total measure on } \Omega \text{ deleted strictly before time } T_1 \text{ is}$$
$$\leq \sum_{n=0}^{\infty} K^n (K+1)(C\delta_1\lambda^{-n})\lambda^{-n} = C(K+1)\delta_1 \sum_{n=0}^{\infty} K^n \lambda^{-2n}.$$

Since $K < \lambda$ by (H4), this number is arbitrarily small as $\delta_1 \to 0$.

Next we estimate the measure deleted between T_k and T_{k+1}. Let ω be a component of Ω_{j-1} with $T_k|\omega = j$. We wish to repeat the argument in the last paragraph with $f^j\omega$ in the place of ω, but can't quite do it because $\ell(f^j\omega) > \bar{\varepsilon}$. So we subdivide ω into $\omega = \omega_1 \cup \cdots \cup \omega_m$ with $\frac{1}{2}\bar{\varepsilon} < \ell(f^j\omega_i) < \bar{\varepsilon}$ and argue that

$$\begin{aligned} &\text{the fraction of } \omega \text{ deleted during the period } [T_k,\ T_{k+1}) \\ \approx\ &\text{the fraction of each } f^j\omega_i \text{ deleted before the next stopping time} \\ \leq\ &2\bar{\varepsilon}^{-1}\sum_{n=0}^{\infty} K^n(K+1)(C\delta_1\lambda^{-(n+j)})\lambda^{-n}. \end{aligned}$$

In "$\approx$" above, we have used the distortion estimate in 7.2 for $f^j|\omega_i$. Since Θ_k is the disjoint union of segments of the type ω, the estimate above carries over to all of Θ_k. That is, we have shown that

$$\begin{aligned} &\text{the total measure of } \Theta_k \text{ deleted during the period } [T_k,\ T_{k+1}) \text{ is} \\ \leq &\frac{C'\delta_1}{\bar{\varepsilon}}\lambda^{-k}\sum_{n=0}^{\infty} K^n\lambda^{-2n}\cdot \ell(\Omega). \end{aligned}$$

Summing over k, we see that the total measure of Ω deleted can be made arbitrarily small by choosing δ_1 small. ∎

We now modify the argument above slightly to suit the setting of Sublemma 1.

Proof of Sublemma 1. Let γ be an arbitrary u-curve, and let $\mu_N := \frac{1}{N}\sum_{i=0}^{N-1} f_*^i\mu_\gamma$. Our invariant measure $\hat{\mu}$ will be a limit point of $\{\mu_N\}_{N=1,2,\ldots}$ normalized.

Here we do not delete any part of γ, but otherwise define T_k and Θ_k as before. Then

$$\left(\sum_{i=0}^{N-1} f_*^i\mu_\gamma\right) U_\varepsilon(S) \leq \int (\tau_0 + \cdots \tau_{N-1})d\mu_\gamma$$

where $\tau_0 := 0$ and

$$\tau_k(x) := \begin{cases} 0 & \text{if } \ x \notin \Theta_k \\ \sharp\{n : f^n x \in U_\varepsilon(S),\ T_k \leq n < t_{k+1}\} & \text{if } \ x \in \Theta_k. \end{cases}$$

To estimate $\int \tau_k$, let $\omega = \omega_1 \cup \cdots \cup \omega_m$ be as in the proof of Sublemma 2 with $T_k \mid \omega = j$, and argue as before that

$$\begin{aligned} &\sum_{n=0}^{\infty} \mu_\omega\{x : f^{j+n}x \in U_\varepsilon(S),\ j+n < T_{k+1}(x)\} \\ \lesssim\ &\frac{2}{\bar{\varepsilon}}\left(\sum_{n=0}^{\infty} K^n(K+1)(C\varepsilon)\lambda^{-n}\right)\cdot \ell(\omega). \end{aligned}$$

Summing over components of Θ_k we obtain $\int \tau_k d\mu_\gamma \leq C'\ell(\gamma)\varepsilon$; hence $\mu_N(U_\varepsilon(S)) \leq C'\ell(\gamma)\varepsilon\ \forall N$. ∎

7.5. Tail estimates for R.

The aim of this subsection is to prove $\mu_{\Omega(x_i)}\{x \in \Lambda^{(i)} : R(x) > n\} \le C\theta^n$ for some $C > 0$ and $\theta < 1$. By (P5), this estimate (with possibly a larger C) will then hold on all $\gamma \in \Gamma^u(x_i)$.

The proof is carried out in 3 steps. Let ω be the f^k-image of a subsegment of $\Omega(x_i)$. We introduce a stopping time T on ω which gives the number of iterates after time k when the $\tilde{\mathcal{P}}_n$-element containing $f^{-k}x$ reaches a certain length. The first step estimates the distribution of this stopping time for $x \in \omega$. When a segment makes a return, say to $\Lambda^{(j)}$, a certain percentage of the segment is absorbed into $\Lambda^{(j)}$, and infinitely many new elements of $\tilde{\mathcal{P}}_n$ corresponding to the gaps of $\Lambda^{(j)}$ are created. The second step deals with the distribution of this stopping time starting from the union of these gaps. The third step combines the results of the first two to estimate the measure of $\{R > n\}$.

Let i be fixed throughout. We let $\Omega = \Omega(x_i)$, and let Ω_n, $\tilde{\Omega}_n$ and $\tilde{\mathcal{P}}_n$ be as in 7.3. For $\omega \subset f^k\Omega$ and $n = 0, 1, \dots$, we define $\omega_n = \omega \cap f^k\Omega_{k+n}$, $\tilde{\omega}_n = \omega \cap f^k\tilde{\Omega}_{k+n}$ and $\tilde{\mathcal{P}}^\omega_n = (f^k\tilde{\mathcal{P}}_{k+n}) \mid \tilde{\omega}_n$.

Step 1. Let ω be the f^k-image of an element of $\tilde{\mathcal{P}}_k$. We define whenever possible for $x \in \omega$ the following stopping time:

$$\begin{aligned} T(x) = {} & \text{the smallest } n > 0 \text{ s.t. the component of } f^n(\tilde{\mathcal{P}}^\omega_{n-1}(x) \cap \omega_n) \\ & \text{containing } f^n x \text{ has length} > 10\delta_0\text{; this assumes in particular that} \\ & x \in \omega_n. \end{aligned}$$

Sublemma 4. *$\exists D_1 > 0$ and $\theta_1 < 1$ s.t. if ω is as above, then $\forall n \ge 1$,*

$$\mu_\omega(\omega_n - \{T \le n\}) \le D_1\theta_1^n.$$

Proof. (Note that the return process to $\cup\Lambda^{(j)}$ is irrelevant in this estimate, for the stopping time T must be reached before any return is possible.) Our hypothesis on ω implies that $\ell(\omega) \le \bar{\varepsilon}$. Reasoning as in 7.4, we see that $\omega_n - \{T \le n\}$ is the union of $\le K^{n+1}$ elements of $\tilde{\mathcal{P}}^\omega_n$ and the f^n-image of each has length $\le 10\delta_0$. Hence $\mu_\omega(\omega_n - \{T \le n\}) \le K^{n+1}(10\delta_0)\lambda^{-n}$. ■

Step 2. In this second step we consider a segment ω contained in the f^k-image of a $\tilde{\mathcal{P}}_{k-1}$-element making a return to $\cup\Lambda^{(j)}$ at time k. Let us assume in fact that ω is stretched exactly across $Q(x_j)$ for some j. Let $\mathcal{G} = \{$ subsegments of ω corresponding to the gaps of $\Lambda^{(j)}$ $\}$, and let $\omega^c := \omega - \Lambda^{(j)} = \bigcup_{\omega' \in \mathcal{G}} \omega'$. Let $T : \omega^c \to \mathbb{Z}^+$ be defined as in Step 1, using ω' in the definition of $T(x)$ for $x \in \omega' \in \mathcal{G}$.

Sublemma 5. *$\exists D_2 > 0$ and $\theta_2 < 1$ independent of ω s.t. $\forall n \ge 1$,*

$$\mu_\omega(\omega^c_n - \{T \le n\}) \le D_2\theta_2^n.$$

We define the *generation* of $\omega' \in \mathcal{G}$ to be the time of its creation. More precisely, let $\hat{\omega}'$ be the subsegment of $\Omega(x_j)$ corresponding to ω', i.e. they straddle the same gap. Then $\text{gen}(\omega') = q$ if q is the first time part of $\hat{\omega}'$ is removed in the construction of $\Omega_\infty(x_j)$. We do not preclude the possibility that this initial break in Ω is enlarged at a later stage. Let $\mathcal{G}_q := \{\omega' \in \mathcal{G} : \text{gen}(\omega') = q\}$.

Proof of Sublemma 5. Our strategy is to majorize $\mu_\omega(\omega_n^c - \{T \le n\})$ by (I) + (II) where

$$\text{(I)} := \sum_{q>\varepsilon n} \sum_{\omega' \in \mathcal{G}_q} \ell(\omega'),$$

$$\text{(II)} := \sum_{q=1}^{\varepsilon n} \sum_{\omega' \in \mathcal{G}_q} \mu_\omega(\omega'_n - \{T \le n\})$$

and $\varepsilon > 0$ is a small number to be determined.

Sub-sublemma. *Let $\omega' \subset \omega$ be a gap of generation q. Then $f^q\omega'$ has only one connected component and has length $\ge \frac{1}{2}\delta_1\lambda^{-q}$.*

Proof. We consider the segment $\hat{\omega}' \subset \Omega(x_j)$ that corresponds to ω'. $\text{Gen}(\hat{\omega}') = q$ implies, by definition, that $\hat{\omega}' \subset \Omega_{q-1}$, so that $f^q\hat{\omega}'$ has only one component. It also implies that the segment deleted at time q does not run into previously created gaps, i.e. $d(f^q(\partial(\text{segment deleted at time } q)),\ S) = \delta_1\lambda^{-q}$. The proof now uses the small alteration in the definition of $\Omega_\infty(x_j)$ made after the statement of Sublemma 2, namely that there exists $\hat{x} \in \hat{\omega}'$ s.t. $d(f^q\hat{x},\ S) < \frac{1}{2}\delta_1\lambda^{-q}$. Therefore we have $d(f^q\partial(\text{deleted part}),\ f^q\hat{x}) \ge \frac{1}{2}\delta_1\lambda^{-q}$, proving $\ell(f^q\hat{\omega}') \ge \delta_1\lambda^{-q}$. Recalling the picture described in Fact (1) of 7.3, we see that $f^q\omega'$ is roughly parallel to $f^q\hat{\omega}'$ with the Hausdorff distance between them $\le \delta\lambda^{-q}$. Thus $\ell(f^q\omega') \approx \ell(f^q\hat{\omega}')$. ∎

To estimate (I), we fix n for now and suppose first that $\omega = \Omega(x_j)$. From Sublemma 2, we see that the total measure of all the segments deleted at or after time q is $\le CK^q\lambda^{-2q}$. Thus (I) $\le C\lambda^{-\varepsilon n}$. Now for $\omega \ne \Omega(x_j)$, which is usually the case, the last line of the last paragraph together with standard distortion estimates allow us to conclude that $\ell(\omega') \approx \ell(\hat{\omega}')$ for all $\omega \in \mathcal{G}$.

To estimate (II), we fix $q \le \varepsilon n$ and consider $\omega' \in \mathcal{G}_q$. Our plan is to apply Sublemma 4 to $f^{q-1}\omega'$. ¿From above we see that $\omega' = \omega'_{q-1}$ for $1 \le i \le q$. If $\ell(f^i\omega') > 10\delta_0$ for some $i < q$, then $T|\omega' < q$ and there is no need to consider ω'. If not, then $f^{q-1}\omega'$ satisfies the hypothesis of Step 1. Applying Sublemma 4 to $f^{q-1}\omega'$ and pulling back, we obtain

$$\begin{aligned} \mu_\omega(\omega'_n - \{T \le n\}) &\le C \cdot \frac{\ell(\omega')}{\ell(f^{q-1}\omega')} \cdot D_1\theta_1^{n-q+1} \\ &= C\ell(\omega') \cdot \frac{\ell(f^q\omega')}{\ell(f^{q-1}\omega')} \cdot \frac{1}{\ell(f^q\omega')} \cdot D_1\theta_1^{n-q+1} \\ &\le C\ell(\omega') \cdot \max|Df| \cdot 2\delta_1^{-1}\lambda^q \cdot D_1\theta_1^{n-q+1}. \end{aligned}$$

Choosing $\varepsilon > 0$ small enough that $\theta_2' := \lambda^\varepsilon \theta_1^{1-\varepsilon} < 1$, we obtain that

$$\sum_{q=1}^{\varepsilon n} \sum_{\omega' \in \mathcal{G}_q} \mu_\omega(\omega_n' - \{T \leq n\}) \leq C' \delta_1^{-1} (\theta_2')^n \left(\sum_{\omega' \in \mathcal{G}} \ell(\omega') \right) \leq C''(\theta_2')^n.$$

Let $\theta_2 = max\{\theta_2', \lambda^{-\varepsilon}\}$. ■

Step 3. For $x \in \Omega$, let $E(x)$, the time of ejection, be the largest n s.t. $x \in \Omega_n$; in particular, $E = \infty$ on Ω_∞. We introduce a sequence of stopping times $T_1 < T_2 < \cdots$ on subsets of Ω as follows. Say T_1 is defined at x if $\exists n > R_1$ s.t. $E(x) \geq n$ and the component of $f^n(\tilde{\mathcal{P}}_{n-1}(x) \cap \Omega_n)$ containing $f^n x$ has length $> 10\delta_0$; $T_1(x)$ is then defined to be the smallest such n. Note that according to our definition of f^R in 7.3, $T_1(x)$ is exactly the first time when part of the component of $f^n(\tilde{\mathcal{P}}_{n-1}(x) \cap \Omega_n)$ containing $f^n x$ returns to $\cup \Lambda^{(i)}$. Let $\Theta_1 = \{x : T_1(x) \text{ is defined}\}$. We consider $x \in \Theta_1 - \{R = T_1\}$ and say T_2 is defined at x if $\exists n > T_1(x)$ s.t. $E(x) \geq n$ and the component of $f^n(\tilde{\mathcal{P}}_{n-1}(x) \cap \Omega_n)$ containing $f^n x$ has length $> 10\delta_0$, and so on. In general, we let $\Theta_k = \{T_k \text{ is defined}\}$ and attempt to define T_{k+1} on $\Theta_k - \{R = T_k\}$.

We observe that for $\mu_\Omega - a.e.\ x \in \Omega_\infty$, either $R(x) < \infty$ or $x \in \Theta_k\ \forall k$. This is because T_1 is defined for $\mu_\Omega - a.e.\ x \in \Omega_\infty$ (Sublemma 4) and also that for each k, T_{k+1} is defined for $\mu_\Omega - a.e.\ x \in \Omega_\infty \cap (\Theta_k - \{R = T_k\})$. This combined with the estimate on $\mu_\Omega(\Theta_k)$ below shows that $R < \infty$ $a.e.$ on Ω_∞.

Now there exists $\varepsilon_1 > 0$ s.t. if ω is a component of $\tilde{\mathcal{P}}_{n-1}(\cdot) \cap \Omega_n$ part of which returns at time n, then

$$\frac{\mu_\Omega(\omega \cap \{R = n\})}{\mu_\Omega(\omega)} > \varepsilon_1.$$

This implies that

$$\frac{\mu_\Omega(\Theta_k \cap \{E \geq T_k \text{ and } R \neq T_k\})}{\mu_\Omega(\Theta_k)} < 1 - \varepsilon_1;$$

hence

$$\mu_\Omega(\Theta_k) \leq (1 - \varepsilon_1)^k$$

which says in particular that

$$\mu_\Omega\{x \in \Omega_\infty : R > T_k\} \leq (1 - \varepsilon_1)^k.$$

For all $n,\ k \in \mathbb{Z}^+$, we have on Ω_∞:

$$\{R > n\} \ \subset \ \{T_k > n\} \cup \{T_k \leq n \text{ and } R > T_k\}.$$

To finish, then, it suffices to prove

Sublemma 6. $\exists D_3 > 0,\ \theta_3 < 1$ *and* $\varepsilon' > 0$ *s.t.* $\forall n \geq 1$,

$$\mu_\Omega\{x \in \tilde{\Omega}_n : T_{[\varepsilon' n]}(x) > n\} \leq D_3\theta_3^n.$$

Proof (a similar argument is used in [C1]). Let $1 \leq k_1 < k_2 < \cdots < k_p \leq n$ be fixed for the time being. For $k \leq n$, we let $A_k := \{x \in \tilde{\Omega}_k : T_i(x) = k_i \ \forall k_i \leq k\}$ and estimate the measure of A_k as follows:

(i) Assume $k_1 > R_1$, otherwise $\mu_\Omega(A_k) = 0$. Since there are $\leq C$ elements in $\tilde{\mathcal{P}}_{R_1}$, $\mu_\Omega(A_{k_1-1}) \leq C\lambda^{-R_1}D_1\theta_1^{k_1-1-R_1}$ by applying Sublemma 4 to $\omega =$ the f^{R_1}-image of each element of $\tilde{\mathcal{P}}_{R_1}$.

(ii) Let ω be a component of $A_{k_1-1} \cap \Omega_{k_1}$ with $T_1 \mid \omega = k_1$. We write $\omega - \{R = k_1\} = (\cup\omega_r') \cup (\cup w_r'')$ where $f^{k_1}\omega_r'$ is the union of gaps of some $\Lambda^{(j_r)}$ and $f^{k_1}\omega_r''$ consists of 2 segments between $2\delta_0$ and $10\delta_0$ in length sticking out on each side of $Q(x_{j_r})$. Applying Sublemma 5 to $f^{k_1}\omega_r'$ and pulling back, we obtain

$$\begin{aligned}\mu_\Omega(\omega_r' \cap A_{k_2-1}) &\leq \frac{C\ell(\bar{\omega}_r')}{\ell(f^{k_1}\bar{\omega}_r')} \cdot D_2\theta_2^{k_2-k_1-1} \\ &\leq \frac{C'\ell(\bar{\omega}_r')}{2\delta_0}D_2\theta_2^{k_2-k_1-1}\end{aligned}$$

where $\bar{\omega}_r'$ is the shortest segment containing ω_r'. Sublemma 4 gives a similar estimate for ω_r''. Combining, we have

$$\frac{\mu_\Omega(A_{k_2-1})}{\mu_\Omega(A_{k_1-1})} \leq \frac{\mu_\Omega(A_{k_2-1})}{\mu_\Omega(A_{k_1})} \leq D_3'(\theta_3')^{k_2-k_1-1}$$

where $\theta_3' = \max(\theta_1, \theta_2)$ and D_3' is independent of k_i or ω.

(iii) Repeating (ii) from k_{i-1} to k_i, $i = 2, \ldots, p$, we obtain

$$\begin{aligned}\mu_\Omega(A_n) = &\ \frac{\mu_\Omega(A_n)}{\mu_\Omega(A_{k_p-1})} \cdot \frac{\mu_\Omega(A_{k_p-1})}{\mu_\Omega(A_{k_{p-1}-1})} \cdot \cdots \cdot \frac{\mu_\Omega(A_{k_2-1})}{\mu_\Omega(A_{k_1-1})} \cdot \mu_\Omega(A_{k_1-1}) \\ &\leq C\left(\frac{D_3'}{\theta_3'}\right)^p \cdot (\theta_3')^n.\end{aligned}$$

Choosing ε' small enough that

$$\theta_3'' := (D_3'\theta_3'^{-1})^{\varepsilon'}\theta_3' < 1,$$

we conclude that

$$\begin{aligned}\mu_\Omega\{x \in \tilde{\Omega}_n : T_{[\varepsilon' n]} > n\} &\leq \sum_{p=0}^{[\varepsilon' n]} \sum_{(k_1,\ldots,k_p):1\leq k_1\leq\cdots\leq k_p\leq n} \mu_\Omega(A_n(k_1,\ldots,k_p)) \\ &\leq C\sum_{p=0}^{[\varepsilon' n]} \binom{n}{p}(\theta_3'')^n\end{aligned}$$

which is $< D_3\theta_3^n$ by Sterling's formula provided ε' is sufficiently small. ∎

7.6. Proofs of theorems.

Constructing a tower $F : \Delta \circlearrowleft$ over $f^R : \underset{i}{\cup}\Lambda^{(i)} \circlearrowleft$ as in Part I and using the fact that $\int R d\mu_\gamma < \infty$ (where γ in this subsection denotes an arbitrary curve in Γ^u), one constructs an F-invariant measure $\tilde{\nu}$ on Δ that projects onto an SRB measure on M. We will show that for $\Lambda^* =$ any $\Lambda^{(i)}$ with $\tilde{\nu}\Lambda^{(i)} > 0$, if $f^{R^*} : \Lambda^* \circlearrowleft$ is the first return map to Λ^* under f^R, then not only do we have $\int R^* d\mu_\gamma < \infty$ but also $\mu_\gamma\{R^* > n\} < C_* \theta_*^n$ for some $\theta_* < 1$. This will prove that f fits (in the strictest sense) the model of Part I; the exponential mixing results will also follow.

Let $\Lambda^{(1)}, \ldots, \Lambda^{(k)}$ be the hyperbolic product sets constructed earlier on in this section. Renumbering, let $\Lambda^* = \Lambda^{(1)}, \ldots, \Lambda^{(q)}$ be s.t. $\underset{i \le q}{\cup} \Lambda^{(i)}$ has positive $\tilde{\nu}$-measure and is f^R-invariant and irreducible (irreducibility means that $\forall i, j \le q$, $\exists n = n(i,j)$ s.t. $(f^R)^n \Lambda^{(i)} \cap \Lambda^{(j)}$ has positive $\tilde{\nu}$-measure.) The $\Lambda^{(i)}$'s, $i > q$, will be discarded from here on.

We introduce on $\cup\Lambda^{(i)}$ the following stopping times: $S_0 \equiv 0$, and $S_k(x) := S_{k-1}(x) + R(f^{S_{k-1}}x)$. Let ξ be the partition of $\cup\Lambda^{(i)}$ into s-subsets $\Lambda_m^{(i)}$ s.t. $f^R\Lambda_m^{(i)}$ is a u-subset of some $\Lambda^{(j)}$, and let $\xi_k := \xi \vee f^{-S_1}\xi \vee \cdots \vee f^{-S_{k-1}}\xi$. Then f^{S_k} maps each element of ξ_k onto a u-subset of some $\Lambda^{(j)}$, and $f^{S_k} \mid (\gamma \cap \xi_k(\cdot))$ has uniform distortion. These two facts will be relied upon heavily in the proofs of the following claims:

(i) $\exists N \in \mathbb{Z}^+$ and $\varepsilon > 0$ s.t. $\forall k$, $\mu_{\gamma\cap\Lambda^*}\{R^* > S_{kN}\} \le (1-\varepsilon)^k$. This is because $f^R : \cup\Lambda^{(i)} \circlearrowleft$ behaves like an irreducible finite state Markov chain.

(ii) $\exists \varepsilon' > 0$ s.t. $\mu_{\gamma\cap\Lambda^*}\{S_{[\varepsilon' n]} > n\} < D_3\theta_3^n$ $\forall n$. The proof is the same as that in Sublemma 6; it uses the fact that $\forall x,\ k,\ \mu_{\gamma\cap\xi_k(x)}\{S_{k+1} - S_k > n\} < C\theta^n$, which is precisely the tail estimate for R.

The desired estimate for $\mu_\gamma\{R^* > n\}$ follows from (i) and (ii).

8. Billiards with Convex Scatterers

8.1. Results and discussions.

The purpose of this section is to illustrate how the model in Part I is applicable to a class of billiards. We will not attempt to include as large a class as possible, but will focus only on billiards bouncing off convex scatterers on the 2-torus. More precisely, let $\{\Gamma_i, i = 1, \ldots, d\}$ be pairwise disjoint C^3 simply connected curves on $\mathbb{T}^2$ with strictly positive curvature, and consider billiards on the domain $X := \mathbb{T}^2 - \underset{i}{\cup}\{\text{interior } \Gamma_i\}$. We assume the "finite horizon" condition, i.e. there is an upper bound on how many consecutive times a billiard trajectory can meet $\cup\Gamma_i$ tangentially. Let $M = \cup\Gamma_i \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ be the usual cross-section in the phase space of the billiard flow, and let $f : M \circlearrowleft$ be the Poincaré map. The coordinates on M are denoted by (r, φ) where $r \in \cup\Gamma_i$ is parametrized by arc length and φ is the angle a unit tangent vector at r makes with the normal pointing into the domain X. It is well known that f preserves the measure $\nu = c\cos\varphi dr d\varphi$ where c is the normalizing constant, and it has been proved in [S2] that (f^n, ν) is ergodic for all $n \geq 1$.

Theorem 6. *Let f be the billiard map above. Then*
(a) f fits the model of Part I with an exponential estimate for $\{R > n\}$;
(b) (f, ν) has exponential decay of correlations and CLT.

"Subexponential" or "stretched exponential" decay of correlations and CLT are proved in [BCS2] for a class of hyperbolic billiards larger than ours. We do not claim the following, but remark that with suitable technical modifications our proof is likely to give exponential mixing for scattering billiards on domains in $\mathbb{R}^2$ with piecewise smooth boundaries – provided the boundary components are "in general position" and intersect with positive angles (see [BSC2]).

For the type of billiards we are considering, $f : M \circlearrowleft$ is essentially a piecewise hyperbolic map in the sense of Section 7. Deferring precise statements to the next subsection, we mention here three of the biggest differences between the present setting and the previous one:

(1) due to the nature of near-tangential reflections, Df here is not bounded; this leads to more complicated distortion estimates and other technical difficulties;
(2) billiards are conservative systems, and with respect to its natural invariant measure ν, f has been shown to be mixing;
(3) Condition (H4) in 7.1 is always valid for this class of billiards.

As pointed out by Chernov and Sinai, the methods of this paper will most likely apply also to Sinai billiards in small external fields with Gaussian thermostats [CELS], although some technical modifications of the proofs are needed due to the fact that these dynamical systems are not conservative (and their derivatives are not uniformly bounded as assumed in Section 7).

We remark that all of the basic technical work associated with (1) and (2) above has been done by Sinai, Bunimovich and Chernov; a concise summary of it is given in [BSC2]. The reasons for (3) are also observed by Bunimovich and noted in [BSC1]. We will give a brief review in 8.2 of what we use, referring to the above mentioned papers for proofs, and then proceed to verify details that are more specific to our constructions.

8.2. Background on billiards.

For the convenience of the reader we summarize below some elementary facts about our class of billiards. Where no additional references are given, the material in this subsection is taken from [BSC2], Sections 2.1 and 2.2. For more information see [BSC2] and the references therein.

A. The discontinuity sets. Let $S_0 = \partial M = \cup \Gamma_i \times \left\{\pm \frac{\pi}{2}\right\}$. Then $f^{-1}S_0$ is the discontinuity set of f. It is easy to see that $f^{-1}S_0$ is the union of a finite number of smooth segments, and that f maps each component of $M - (S_0 \cup f^{-1}S_0)$ diffeomorphically onto its image. All the curves in $f^{-1}S_0$ have negative slopes which are bounded away from 0 and $-\infty$. (See B. below.) More generally, for $n \geq 1$, the discontinuity set of f^n, namely $\bigcup_{i=1}^{n} f^{-i}S_0$, consists of a finite number of smooth segments, all negatively sloped. We say that $x \in M$ is an *n-multiple point* if there is more than one smooth curve segment from $\bigcup_{i=1}^{n} f^{-i}S_0$ passing through or ending in x. The following fact is key to proving (H4):

(*) ([BSC1], Lemma 8.4) *the number of curves in $\bigcup_{1}^{n} f^{-i}S_0$ passing through or ending in any one point in M is $\leq K_0 n$, where K_0 is a constant depending only on f.*

We will explain how (*) implies (H4) after paragraph C.

B. Invariant cones and hyperbolicity. Identifying the tangent space at each point with the (r,φ)-plane, Df maps the cone $\{r\varphi \geq 0\}$ strictly into itself with the image uniformly bounded away from the r- and φ-axes. Let us call the Df-image of $\{r\varphi \geq 0\}$ the *unstable cone* C^u. Similarly, Df^{-1} maps $\{r\varphi \leq 0\}$ strictly into itself and C^s is defined accordingly. One could prove hyperbolicity directly ([S2], see C. below), or conclude using [W] and [KS] that there are nonzero Lyapunov exponents and stable and unstable manifolds a.e. Note that all unstable curves have positive slopes bounded away from 0 and ∞. Also, the angles between W^u- and W^s-curves as well as those between W^u-curves and the discontinuity curves in $\bigcup_{i=1}^{n} f^{-i}S_0$ are bounded away from 0.

C. The "p-metric". Expansion and contraction coefficients are more conveniently described in terms of the semi-metric $\cos\varphi dr$, which has geometric meaning and which we will loosely call "the p-metric". Norms with respect to this semi-metric are denoted by $|\cdot|_p$, as opposed to $|\cdot|$ which refers to the Euclidean metric. We will write $p(\cdot)$ for the p-length of a curve, while $\ell(\cdot)$ denotes its Euclidean length and $d(\cdot,\cdot)$ denotes Euclidean distance.

Facts. 1. (a) if $v \in T_xM$ is such that $v \in C^u$ or C^s, then

$$|v|_p \approx |v| \cdot d(x, S_0) \quad ^{1} \;;$$

(b) if γ is a u-curve (i.e. all of its tangent vectors are in C^u) or an s-curve, then

$$C_1 p(\gamma) \leq \ell(\gamma) \leq C_2 \sqrt{p(\gamma)}$$

for some C_1, C_2 depending only on f.

2. (a) $\exists\ \lambda > 1$ s.t.

$$|Dfv|_p \geq \lambda |v|_p \quad \forall v \in C^u,$$

[1] The symbol "$\approx$", which will be used many times in this section, has the following precise meaning: "$A \approx B$" means that there is a $c > 0$ depending only on f and on other universal constants s.t. $c^{-1}A \leq B \leq cA$. Also, "$A \gtrsim B$" is to be interpreted accordingly.

$$|Df^{-1}v|_p \geq \lambda |v|_p \quad \forall v \in C^s.$$

(b) for $v \in T_xM$ with $v \in C^u$ and $|v|_p = 1$,

$$|Df_x v|_p \approx \frac{1}{d(x, S_0)} \ ;$$

in particular, $|Df_x v|_p \to \infty$ as $x \to \partial M$.

Justification of (H4). Because of the geometry of the singularity set, it suffices here to have (H4) with $\delta_i = 0$ (see 8.4 for an explanation). Since n-multiple points are isolated, all sufficiently short u-curves in sufficiently small neighborhoods of them intersect $\bigcup\limits_1^n f^{-i} S_0$ transversally in $\leq K_0 n$ points, i.e. $f^n \gamma$ has $\leq K_0 n + 1$ connected components. There clearly is no problem away from multiple points. Thus it suffices to choose N s.t. $K_0 N + 1 < \lambda^N$. (As we shall see, it is the expansion coefficient wrt the p-metric that we want.)

D. "Homogeneity strips" and distortion control. Following [BSC2], Section 3, we subdivide a neighborhood of S_0 or ∂M into strips on which derivatives are roughly comparable. Fix $k_0 \in \mathbb{Z}^+$ sufficiently large. For $k \geq k_0$, let

$$I_k = \left\{ (r, \varphi) : \frac{\pi}{2} - \frac{1}{k^2} < \varphi \leq \frac{\pi}{2} - \frac{1}{(k+1)^2} \right\},$$

and

$$I_{-k} = \left\{ (r, \varphi) : -\frac{\pi}{2} + \frac{1}{(k+1)^2} \leq \varphi < -\frac{\pi}{2} + \frac{1}{k^2} \right\}.$$

We say that a local unstable manifold γ is *homogeneous* if for all $n \geq 0$, $f^{-n}\gamma$ is contained in three adjacent I_k's. (For convenience of language, let us agree to call $M - \bigcup\limits_{|k| \geq k_0} I_k$ one of the I_k's also.) The homogeneity of a piece of local stable manifold is defined similarly.

Homogeneous stable and unstable manifolds are known to exist a.e.; we will prove a version of this result that suits our purposes in the next subsection. Because of the unboundedness of the derivatives, distortion along W^u_{loc} and the Jacobians of the maps between W^u_{loc}'s obtained from sliding along W^s_{loc}'s do not have uniform estimates unless we restrict ourselves to homogeneous W^u_{loc}'s and W^s_{loc}'s. We will return to this when verifying (P3)-(P5).

8.3. Construction of $f^R : \Lambda \circlearrowleft$ and verification of (P1)-(P5).

When working with distances on W^s_{loc}- and W^u_{loc}-curves, we will use exclusively the p-metric. In particular, $W^u_\delta(x)$ denotes a piece of W^u_{loc}-curve with $p(W^u_\delta(x)) = \delta$, and $(f|\gamma)'$ denotes the derivative of f with respect to the p-metric.

Let $\lambda_1 = \lambda^{\frac{1}{4}}$ where $\lambda > 1$ is as in 8.2, and let $\delta_1 > 0$ be a small number. A finite number of conditions will be imposed on δ_1 as we go along. Let

$$B^+_{\lambda_1, \delta_1} = \{x \in M : d(f^n x,\ S_0 \cup f^{-1} S_0) \geq \delta_1 \lambda_1^{-n}\ \forall n \geq 0\},$$

$$B^-_{\lambda_1, \delta_1} = \{x \in M : d(f^{-n} x,\ S_0 \cup f S_0) \geq \delta_1 \lambda_1^{-n}\ \forall n \geq 0\},$$

and let $\delta = \delta_1^4$.

Sublemma 1. (i) $\forall x \in B^+_{\lambda_1,\delta_1}$, *$W^s_{10\delta}(x)$ exists and is homogeneous.*

(ii) *The corresponding statement about $W^u_{10\delta}(x)$ holds for $x \in B^-_{\lambda_1,\delta_1}$.*

Proof. First we assume $W^s_{10\delta}(x)$ exists and prove homogeneity. Let $f^n x \in I_k$. Then $\frac{1}{k^2} \geq \delta_1 \lambda_1^{-n}$, and the (Euclidean) width of I_k is $\approx \frac{1}{k^3}$, which is $\geq (\delta_1 \lambda_1^{-n})^{\frac{3}{2}}$. But since $p(f^n W^s_{10\delta}(x)) \leq 10\delta\lambda^{-n}$, $\ell(f^n W^s_{10\delta}(x)) \lesssim 10(\delta_1\lambda_1^{-n})^4 \cdot k^2 \leq 10(\delta_1\lambda_1^{-n})^3$.

The construction of W^s-curves is standard and is sketched in 7.3, but let us verify that there are no new difficulties in the present situation. Let γ be an s-curve centered at $f^n x$ having p-length $20\delta\lambda^{-n}$. First we claim that $\gamma \cap fS_0 = \phi$. Suppose not. Let $\tilde{\gamma}$ be the subsegment of γ joining $f^n x$ to fS_0. We may assume $\tilde{\gamma}$ meets a branch $\tilde{S}$ of fS_0 in a point that is not a multiple point. Now $\tilde{S}$ has two f^{-1}-images depending on the side from which one approaches it: one lies in S_0 and the other in $f^{-1}S_0$. In either case, $\ell(f^{-1}\tilde{\gamma}) \geq \delta_1 \lambda_1^{-(n-1)}$ because $d(f^{n-1}x, S_0 \cup f^{-1}S_0) \geq \delta_1\lambda_1^{-(n-1)}$. From the estimates in the last paragraph, on the other hand, we have that $\tilde{\gamma}$ is contained in two contiguous $I'_k s$, and by Fact 2(b) in 8.2C applied to $f^{-1}, (f^{-1}|\tilde{\gamma})' \approx k^2 \leq (\delta_1\lambda_1^{-n})^{-1}$, giving $p(f^{-1}\tilde{\gamma}) \lesssim (\delta_1\lambda_1^{-n})^3$ which is not compatible with the ℓ-length estimate above.

We have shown that $\gamma \cap fS_0 = \phi$, which means that $f^{-1}\gamma$ is connected with $p(f^{-1}\gamma) \geq \lambda p(\gamma) \geq 20\delta\lambda^{-(n-1)}$. Continuing to iterate backwards we see that the component of $f^{-n}\gamma$ containing x has p-length $\geq 20\delta$. ■

By the Borel-Cantelli Lemma, $\nu B^+_{\lambda_1,2\delta_1}$ and $\nu B^-_{\lambda_1,2\delta_1}$ are ≈ 1 if δ_1 is sufficiently small. Let x_1 be a Lebesgue density point of $B^+_{\lambda_1,2\delta_1} \cap B^-_{\lambda_1,2\delta_1}$ chosen away from $f^{-1}S_0 \cup S_0 \cup fS_0$. We construct Λ around x_1 as follows: Let $\Omega = W^u_\delta(x_1)$. For $n \geq 0$, let

$$\Omega_n = \{y \in \Omega : d(f^i y, f^{-1}S_0) \geq \delta_1\lambda_1^{-i},\ i = 0, 1, ..., n$$

$$\text{and } d(f^i y, S_0) \geq \delta_1\lambda_1^{-i},\ i = 0, 1, ..., n+1\}.$$

In particular, if ω is a component of Ω_n, then $f^{n+1}\omega$ is connected and $d(f^{n+1}\omega, S_0) \geq \delta_1\lambda_1^{-(n+1)}$. (The second condition in the definition of Ω_n is redundant most of the time.) Let $\Omega_\infty = \bigcap_n \Omega_n$. By Sublemma 1, $W^s_\delta(y)$ exists and is homogeneous for all $y \in \Omega_\infty$. Let $\Gamma^s = \{W^s_\delta(y) : y \in \Omega_\infty\}$, $\Gamma^u = \{$all homogeneous W^u_{loc}-curves that meet every $\gamma^s \in \Gamma^s$ and which stick out by $> \delta$ on both sides of the curves in $\Gamma^s\}$, and let Λ be the hyperbolic product set defined by Γ^u and Γ^s. As in 7.3, let Q be a rectangular shaped region containing Λ.

First we consider the measure of Λ. Since x_1 is a density point of $B^+_{\lambda_1,2\delta_1}$, there is a ν-positive measure set of points in Q through which $W^s_{10\delta}$-curves exist. Assuming the absolute continuity of homogeneous W^s_{loc}-curves (see (P5) below), these curves meet Ω in a set of positive measure, and points in the intersection clearly belong in Ω_∞. This proves (P1). In fact, our choice of x_1 ensures the abundance of curves in Γ^u as well, proving $\nu\Lambda > 0$. As we shall see, it follows from our *a priori* knowledge of the mixing property of (f, ν) that this arbitrary choice of Λ will suffice.

We now proceed to verify (P3)-(P5). Since we are using exclusively the p-metric on W^u- and W^s-curves, *all distances and derivatives in* (P3)-(P5)*are wrt the p-metric.* For example, (P5)(b) is about the Radon-Nikodym derivative of the image of one p-measure wrt another, where p-measure on $\gamma^u \in \Gamma^u$ is the measure induced from the p-metric. (P3) is noted in 8.2C. For $x, y \in \gamma^u \in \Gamma^u$, recall that $s_0(x,y) = n$ if $f^i x$ and $f^i y$ are viewed as "together" for $i \leq n$ while $f^{n+1}x$ and f^{n+1} are "separated". In the present setting it is natural to interpret these notions as follows: let $[x,y]$ denote the subsegment of γ^u between x and y. Then $s_0(x,y) = n$ if for every $i \leq n$, $f^i[x,y]$ is a connected smooth curve completely contained in 3 adjacent I_k's, and that either $f^{n+1}[x,y]$ has more than one component or it meets four I_k's (see 7.3). With this interpretation, (P4)(a) is valid because of the backward contraction constant of λ^{-1} (see 8.1 C.) and the fact that there is a finite upper bound to the p-lengths of all connected (homogeneous) W^u_{loc}-curves. (P4)(b) is Proposition A1.1(d) in Appendix 1 of [BSC2] for backward iterates on unstable manifolds. (P5)(a) is part (d) of the same Proposition, and (P5)(b) follows from (P5)(a) and standard hyperbolic arguments (see [BSC2] and also [KS]).

It remains to define $f^R : \Lambda \circlearrowleft$ and to verify (P2). As before, let $\tilde{\Omega}_n = \Omega_n - \{R \leq n\}$ and let $\tilde{\mathcal{P}}_n$ be a partition of $\tilde{\Omega}_n$ whose elements are thought of as representing distinct trajectories. For reasons to become clear in the next subsection, we will allow returns to Λ only at times that are multiples of an integer m to be determined. We describe the procedure below.

First we choose $R_1 > R_0$ large enough that if a component ω of Ω_n u-crosses the middle half of Q under f^n for $n \geq R_1$, then the entire s-subrectangle of Q associated with ω u-crosses Q under f^n.

Let $n \in \mathbb{Z}^+$ be a multiple of m. We assume $\tilde{\Omega}_{n-m}$ and $\tilde{\mathcal{P}}_{n-m}$ are given and define $\tilde{\Omega}_n$ and $\tilde{\mathcal{P}}_n$. But first let us introduce some intermediate partitions $\tilde{\mathcal{P}'}_{n-m+i}$ on $\tilde{\Omega}_{n-m} \cap \Omega_{n-m+i}$ for $i = 1, 2, ..., m$. Let ω be a component of $\tilde{\Omega}_{n-m} \cap \Omega_{n-m+1}$. Then $f^{n-m+1}\omega$ is a smooth curve. We partition ω into segments so that their f^{n+m-1}-images are homogeneous, but will introduce new partition points only when needed, i.e. if a partition point q is introduced and $f^{n-m+1}q \in I_k$, then the f^{n-m+1}-images of both of its adjacent segments must have real length $\approx O(\frac{1}{k^3})$. This defines $\tilde{\mathcal{P}'}_{n-m+1}$. Next we consider each component of $\omega' \cap \Omega_{n-m+2}$ where $\omega' \in \tilde{\mathcal{P}'}_{n-m+1}$, and add partition points of $\tilde{\mathcal{P}'}_{n-m+2}$ where necessary so that their f^{n-m+2}-images are homogeneous, and so on. For $n < R_1$ we let $\tilde{\mathcal{P}}_{n-m+i}$ be the join of $\tilde{\mathcal{P}}_{n-m+i}$ with the components of Ω_{n-m+i} for all $i \leq m$. For $n \geq R_1$, let $\tilde{\mathcal{P}}_{n-m+i}$ be similarly defined for $i < m$, but define $\tilde{\Omega}_n$ and $\tilde{\mathcal{P}}_n$ as follows: consider each $\omega \in \tilde{\mathcal{P}'}_n$. If $f^n\omega$ u-crosses the middle half of Q with $\geq \delta$ sticking out on each side, then we declare that $R = n$ on $\omega \cap f^{-n}\Lambda$, put $\omega - \{R = n\}$ into $\tilde{\Omega}_n$ and let $\tilde{\mathcal{P}}_n \mid (\omega - \{R = n\})$ be the partition into connected components. If $f^n\omega$ does not u-cross Λ in the manner required, then put $\omega \subset \tilde{\Omega}_n$ and let it be an element of $\tilde{\mathcal{P}}_n$.

This procedure defines f^R on part of Ω_∞. To extend the definition to the corresponding part of Λ and to verify (P2), we observe that Sublemma 3 in Section 7 holds with essentially the same proof. Let $\omega \in \tilde{\mathcal{P}'}_n$ be s.t. part of it returns at time n. By design, $f^n\omega$ is homogeneous and is therefore an element of Γ^u. As in 7.3, we consider $y \in f^n\omega \cap \Lambda$, and let $x \in \omega$ be s.t. $f^n x = y$. We need to show that $d(f^j x, S_0 \cup f^{-1}S_0) > \delta_1 \lambda_1^{-j}$ $\forall j > n$. Let y' be the point in Ω_∞ with $y \in W^s_\delta(y')$ and proceed as before observing that

$$\ell(f^{j-n}W^s_\delta(y')) \lesssim \delta\lambda^{-(j-n)}(\delta_1\lambda_1^{-(j-n)})^{-1} = (\delta_1\lambda_1^{-(j-n)})^3$$

so that

$$d(f^j x,\ S_0 \cup f^{-1}S_0) \gtrsim \delta_1\lambda_1^{-(j-n)} > \delta_1\lambda_1^{-j}.$$

This finishes (P2) modulo the a.e. finiteness of R, the proof of which is contained in our estimate of $p\{R > n\}$ in 8.4.

We close with the following remarks on our use of the p-metric on $\gamma^u \in \Gamma^u$ in (P3)-(P5). Referring the reader to 3.1, we observe that there is no internal inconsistency in this reduction argument as long as all the estimates in (P3)-(P5) are wrt the p-metric. In fact, once the reference measure (called $\bar{m}$ there) is defined, the p-measure is forgotten. Caution needs to be exercised, however, when making a connection between $F : \Delta \circlearrowleft$ and test functions on M, the Hölder properties of which are wrt real distance on M. The relevant estimate is contained in the sublemma in 4.1. Using the notation there let γ be a piece of stable or unstable manifold stretched across $\pi F^k M_{2k}(\cdot)$. We have $p(\gamma) \le C\alpha^k$, but need the corresponding estimate for $\ell(\gamma)$. Since γ is homogeneous, we may assume it is contained in some I_n; thus $p(\gamma) \lesssim \frac{1}{n^5}$, and we conclude that $\ell(\gamma) \approx p(\gamma)\cdot n^2 \lesssim p(\gamma)\cdot p(\gamma)^{-\frac{2}{5}}$ which is $< C'\alpha'^k$ for some $\alpha' < 1$.

8.4. Tail estimate for R.

We prove in this subsection that $p\{x \in \Omega_\infty : R(x) > n\} \le C\theta^n$ for some $C > 0$ and $\theta < 1$. The proof follows in outline the 3 steps in 7.5, but some modifications are needed within each step. Recall that returns to Λ are allowed only at time steps that are multiples of an integer m which we now choose. Let $\alpha_0 := 2\sum\limits_{k=k_0}^{\infty}\frac{1}{k^2}$ where $\{I_k, |k| \ge k_0\}$ are the homogeneity strips, and assume that $\lambda^{-1} + \alpha_0 < 1$. Let K_0 be the constant in (*) in 8.2A, and choose m large enough that $(K_0 m + 1)^{\frac{1}{m}}(\lambda^{-1} + \alpha_0) < 1$. Furthermore we let $\bar\varepsilon > 0$ be small enough that every W^u_{loc}-curve of ℓ-length $\le \bar\varepsilon$ has the property that it intersects $\le K_0 m$ smooth segments of $\bigcup\limits_{i=1}^{m} f^{-i}S_0$. For simplicity we take $\bar\varepsilon < \delta$.

Step 1. Let ω be the f^k-image of an element of $\tilde{\mathcal{P}}_k$, where k is a multiple of m. Observe that unlike Section 7, ω here could be quite long compared to δ without any part of it having returned to Λ – although there is an upper bound C_1 on the (ℓ-)lengths of all connected W^u-curves. Let ω_n, $\tilde\omega_n$ and $\tilde{\mathcal{P}}^\omega_n$ be as before, and let $\tilde{\mathcal{P}}'^{\omega}_n$ have the obvious meaning (see 8.3). For $x \in \omega$, define

$$T(x) = \ inf\ \{n > 0\ :\ n \text{ is a multiple of } m \text{ and } \ell(f^n(\tilde{\mathcal{P}}'^{\omega}_n(x))) > \bar\varepsilon\}.$$

Note that since $\bar\varepsilon <$ the width of Λ, T must be reached at or before returns to Λ are possible.

Sublemma 2. *$\exists D_1 > 0$ and $\theta_1 < 1$ independent of ω s.t. $\forall n \geq 1$,*

$$p(\omega_n - \{T \leq n\}) < D_1 \theta_1^n.$$

Proof (following in large part [BSC2], Appendix 3). By (*) in 8.2A and our choice of $\bar{\varepsilon}$, $f^m\omega$ has $\leq C_1\bar{\varepsilon}^{-1}(K_0 m + 1)$ connected components. The following slightly technical point is a consequence of the geometry of $f^{-1}S_0$. We claim that each component of $f^m\omega$ contains at most one component of $f^m\omega_m$. To see this, consider $\gamma =$ the f^i-image of a component of ω_{i-1}, $i \leq m$, and let $\gamma' = \gamma - \{x : d(x,\ f^{-1}S_0) < \delta_1 \lambda_1^{-(k+i)}\}$, i.e. γ' is the part that survives the first "half" of the deletions to take place at the next step. Observe that γ is a connected smooth curve with a strictly positive slope, while all the segments in $f^{-1}S_0$ are negatively sloped. Moreover, every point in $f^{-1}S_0$ lies in a continuous, piecewise smooth segment that extends from $\varphi = -\frac{\pi}{2}$ to $\varphi = \frac{\pi}{2}$. This geometry guarantees that every component of $\gamma - \gamma'$ either contains a point of $\gamma \cap f^{-1}S_0$ or it lies at one of the two ends of γ. Next we delete from γ' the set $\{x : d(fx, S_0) < \delta_1\lambda_1^{-(k+i+1)}\}$. This may trim the edges of some components of γ'; it does not create new components.

Label the connected components of $f^m\omega_m$ as $\gamma^{(1)}, \ldots, \gamma^{(n_1)}$. Each element ω' of $\tilde{\mathcal{P}}'^{\omega}_m$ on ω_m can then be uniquely identified with the $(m+1)$-tuple $(i_1, \ldots, i_m;\ \gamma^{(j_1)})$ where $\gamma^{(j_1)}$ is the component of $f^m\omega_m$ containing $f^m\omega'$ and $f^j\omega'$ is associated with I_{i_j} (write $i_j = *$ if $f^j\omega'$ is associated with the "middle" part of M). If $\ell(f^m\omega') > \bar{\varepsilon}$, then $T \mid \omega' = m$ and we stop considering ω'. If not, we repeat the above with $f^m\omega'$ in the place of ω, and if $f^m\omega' \subset \gamma^{(j_1)}$ we label the components of $f^{2m}((\omega')_{2m})$ as $\gamma^{(j_1 j_2)}$, $j_2 = 1, \ldots, n_{12}$. Proceeding inductively, we conclude that each element of $\tilde{\mathcal{P}}^{\omega}_n \mid (\omega_n - \{T \leq n\})$ can be uniquely associated with an itinerary $i = (i_1, \ldots, i_n;\ \gamma^{(j_1 \cdots j_{n/m})})$ where for each choice of $i_1, \ldots, i_{m(\ell-1)}$ and $j_1, \ldots, j_{\ell-1}$, there are $\leq (K_0 m+1)$ possibilities for j_ℓ. Let us call $\omega(i)$ the segment with itinerary i.

We estimate $p(\omega_n - \{T \leq n\})$ as follows. Remembering that $f' \mid W^u_{loc}$ on I_k is $\approx k^2$, we have for those $\omega(i)$ with $\ell(f^n\omega(i)) \leq \bar{\varepsilon}$

$$p(\omega(i)) \lesssim \frac{Cp(f^n\omega(i))}{(f^n)' \mid \omega(i)} \leq C\bar{\varepsilon}\left(\prod_{j:i_j \neq *} \frac{1}{i_j^2}\right) \cdot \lambda^{-\sharp\{j:i_j=*\}}.$$

Note that this estimate depends only on the $(i_1, ..., i_n)$-part of the itinerary. Summing over all i, we have

$$
\begin{aligned}
p(\omega_n - \{T \le n\}) \;&\lesssim\; C_1\bar{\varepsilon}^{-1}(K_0 m + 1)^{\frac{n}{m}} \sum_{(i_1,\ldots,i_n)} C\bar{\varepsilon}\left(\prod_{j:i_j\neq *}\frac{1}{i_j^2}\right)\cdot \lambda^{-\sharp\{j:i_j=*\}} \\
&\le D_1(K_0 m+1)^{\frac{n}{m}}\sum_{k=0}^{n}\binom{n}{k}\left(\sum_{|\ell|=k_0}^{\infty}\frac{1}{\ell^2}\right)^{k}\lambda^{-(n-k)} \\
&\le\; D_1(K_0 m+1)^{\frac{n}{m}}\sum_{k=0}^{n}\binom{n}{k}\alpha_0^k\lambda^{-(n-k)} \\
&=\; D_1\left[(K_0 m+1)^{\frac{1}{m}}(\alpha_0+\lambda^{-1})\right]^n = D_1\theta_1^n.
\end{aligned}
$$

■

Step 2. As in 7.5 we consider here a subsegment ω of the f^k-image of an element of $\tilde{\mathcal{P}}_{k-1}$ making a return at time k and assume that ω is stretched exactly across Λ. Let ω^c, $\mathcal{G}$, $\mathcal{G}_q$ etc. be as before.

Sublemma 3. *$\exists D_2 > 0$ and $\theta_2 < 1$ independent of ω s.t. $\forall n \ge 1$,*

$$p(\omega_n^c - \{T \le n\}) \le D_2\theta_2^n.$$

Proof. We will estimate (I) and (II) as in the proof of Sublemma 5 of Section 7, but the estimates here need to be handled with more care. Let $\omega' \subset \omega$ be an element of $\mathcal{G}$, and let $\hat{\omega}'$ be the corresponding subsegment of Ω.

Sub-sublemma 1. $p(\omega') \approx p(\hat{\omega}')$

Proof. It follows from the definition of generation and the geometry of $f^{-1}S_0$ that $f^q\hat{\omega}'$ must traverse fully the $\delta_1\lambda_1^{-q}$-neighborhood of some branch of $f^{-1}S_0$ (otherwise the deletion at step q would run into a previously created gap); hence $\ell(f^q\hat{\omega}') \ge 2\delta_1\lambda_1^{-q}$. Also, as part of the requirement for being in $\Omega_{q-1}, \hat{\omega}'$ has the property that $d(f^q\hat{\omega}', S_0) \ge (\delta_1\lambda_1^{-q})$, so $p(f^q\hat{\omega}') \gtrsim (\delta_1\lambda_1^{-q})^2$, which is $>> \delta\lambda^{-q}$. Thus $f^qQ_{\omega'}$ is a long thin rectangular region with its 2 u-sides much longer than its 2 s-sides (see the proof of Sublemma 1 in 8.3). To pull back and conclude that $p(\omega') \approx p(\hat{\omega}')$, we need to subdivide $f^qQ_{\omega'}$ into homogeneous s-subrectangles and compare the u-sides of each. This is necessary because our distortion estimate on W^u-curves works only on homogeneous segments; see (P4)(b). To ensure that the u-sides of these s-subrectangles are comparable, we must have that they continue to be much longer than the corresponding s-sides. This is true because the s-curves joining $f^q\omega'$ to $f^q\hat{\omega}'$ are some order of magnitude shorter than the widths of the $I_k's$ that $f^qQ_{\omega'}$ is allowed to intersect (again see Sublemma 1). ■

Estimation of (I): (Here we do not have an explicit estimate on the measure on Ω deleted after step q; the proof in 7.4 uses more uniform estimates than are available here.) We argue instead that

$$\sum_{q>\varepsilon n} \sum_{\omega' \in \mathcal{G}_q} \nu(Q_{\omega'}) \leq C\sigma^n \tag{$*$}$$

for some $\sigma < 1$. This follows from

(i) the ν-measure of $A_n := \{x \in M : d(f^j x,\ f^{-1}S_0 \cup S_0) < 2\delta_1 \lambda_1^{-j}$ for some $j \geq \varepsilon n\}$ is $< C\sigma^n$; and

(ii) $\forall \omega' \in \mathcal{G}_q$ with $q > \varepsilon n$, $Q_{\omega'} \subset A_n$ (Exercise: show that every $y \in \gamma - \Lambda, \gamma \in \Gamma^u$, can be attached to some $x \in \Omega - \Omega_\infty$ with the property that x is deleted at time j and y lies on a j-stable curve through x).

To pass from (*) to

$$\sum_{q>\varepsilon n} \sum_{\omega' \in \mathcal{G}_q} p(\omega') \leq C\sigma'^n,$$

we use the fact that ν-measure conditioned on γ^u-curves is equivalent to Lebesgue measure and the Sub-sublemma above.

Estimation of (II): Let T be the stopping time introduced in Step 1 starting from ω'. While no deletions are made on ω' in the first $q-1$ steps, $\omega' = \omega'_{q-1}$ can be partitioned into many short segments for reasons of distortion control, and some of them may reach their stopping times before time q. Let $\omega'' \in \tilde{\mathcal{P}'}^{\omega'}_{q-1} \mid (\omega' - \{T < q\})$. Applying Sublemma 2 of this section to $f^{q-1}\omega''$ and pulling back, we obtain

$$\begin{aligned} p(\omega'_n - \{T \leq n\}) &\leq \sum_{\omega''} \frac{Cp(\omega'')}{p(f^{q-1}\omega'')} \cdot D_1 \theta_1^{n-q+1} \\ &\leq Cp(\omega')(\delta_1 \lambda_1^{-q})^{-3} \cdot D_1 \theta_1^{n-q+1}. \end{aligned}$$

The first inequality uses the distortion estimate for $f^{q-1} \mid \omega''$; the second follows from the Sub-sublemma below. The constants ε and θ_2 are then chosen as before.

Sub-sublemma 2. *For all* $\omega'' \in \tilde{\mathcal{P}'}^{\omega'}_{q-1} \mid (\omega' - \{T < q\})$,

$$p(f^{q-1}\omega'') \gtrsim (\delta_1 \lambda_1^{-q})^3.$$

Proof. We argue the following 2 cases differently.

Case 1. $\omega'' \neq \omega'$. Let j be the first moment when ω'' is created. (For example, if both end points of ω'' are partition points of $\tilde{\mathcal{P}'}^{\omega'}_{q-1}$, then j is the time when the second one is inserted.) Recall that with no deletions or returns possible prior to step $q-1$ on $\omega' - \{T < q\}$, all relevant partition points here are introduced only to prevent the violation of homogeneity. Thus we have $f^j\omega'' \approx I_n$ for some n; hence $\ell(f^j\omega'') \gtrsim \frac{1}{n^3} \gtrsim (\delta_1\lambda_1^{-j})^{3/2}$, and $p(f^{q-1}\omega'') \geq p(f^j\omega'') \gtrsim (\delta_1\lambda_1^{-j})^{5/2} \geq (\delta_1\lambda_1^{-q})^{5/2}$.

Case 2. $\omega'' = \omega$, i.e. $\tilde{\mathcal{P}'}^{\omega'}_{q-1}$ consists of a single element. As noted in the proof of Sub-sublemma 1, $p(f^q\hat{\omega}') \geq (\delta_1\lambda_1^q)^2$ and $p(f^q\omega') \approx p(f^q\hat{\omega}')$. We then conclude the desired estimate using $(f|f^{q-1}\omega')' \lesssim \delta_1^{-1}\lambda_1^{q-1}$. ■

Step 3. We will use (without proof) the following fact which is a weak version of Theorem 3.13 in [BSC2]:

(**) *Given $\varepsilon_0 > 0$, $\exists n_0$ s.t. if ω is a homogeneous W^u_{loc}-curve with $p(\omega) > \varepsilon_0$, then $\exists q \leq n_0$ s.t. $f^q\omega$ contains a homogeneous segment which u-crosses the middle half of Q with $> 2\delta$ sticking out from each side.*

For simplicity let us assume for the rest of this proof that $m = 1$. Here is how (**) fits into our scheme. Let T be the stopping time defined in Step 1. As in 7.5, we introduce a sequence of stopping times $T_1 < T_2 < ...$ on Ω with the following properties: let $T_k(x)$ be the time when T is reached for the k^{th} time at x, taking points which have returned to Λ out of circulation as is previously done. We assume also that $T_1 > R_1$, the number we choose to be the first allowable return time. Unlike the set-up in Section 7, however, $T_k|\omega = n$ does not imply that part of ω returns at time n. Instead we have the following: since $f^n\omega$ lies in 3 contiguous I_k's and has ℓ-length $> \bar{\varepsilon}$, it cannot be in $I_k \cup I_{k+1}$ for $k \geq$ some k_1. Thus there exists $\varepsilon_0 > 0$ s.t. $p(f^n\omega) > \varepsilon_0$. Using this ε_0, let n_0 be given by (**), and let $\gamma \subset f^n\omega$ be s.t. $f^q\gamma$ is homogeneous and u-crosses Q as desired.

A technical nuisance: because of our deletions part of γ may be lost in the next q iterates, and we must verify that what remains continues to u-crosse Q with $\geq \delta$ sticking out on each side. First, observe that since $f^q\gamma$ is connected, it follows from the geometry of $f^{-1}S_0$ that all deletions from γ are made at the two ends. Second, $p(f^q\gamma - f^{n+q}\Omega_{n+q}) \leq C\delta_1\lambda_1^{-n} \cdot (f^q|\gamma)'$. This can be made arbitrarily small by proving an upper bound for $(f^q|\gamma)'$ that is independent of n and then choosing R_1 to be suitably large. To prove this bound, suppose that $f^i\gamma$ is contained in I_k for some large k. Then $(f|f^i\gamma)' \approx k^2$ and its image has p-length $\lesssim \frac{1}{k^3}$. Since $f^{i+1}\gamma$ is homogeneous, it is again contained in some $I_{k'}$. Clearly, k' cannot be arbitrarily large in relation to k. Reasoning inductively, we see that there exists k_2 with the property that $\forall i \leq q, f^i\gamma \cap I_k = \phi \ \forall |k| > k_2$, otherwise its length will not be able to grow back to 3δ as required. Hence $(f^q|\gamma)' \lesssim {k_2}^{2n_0}$.

Next we need to argue that there exists $\varepsilon_1 > 0$ s.t. if $\omega \subset \Omega$ is one of the segments with $T_k \mid \omega = n$, then

$$\frac{p(\omega \cap \{R \leq n + n_0\})}{p(\omega)} \geq \varepsilon_1.$$

This requires a distortion estimate for $(f^n|\omega)'$ and a uniform bound on $(f^q|f^n\omega)'$ both of which we have. To estimate $p\{R > n\}$, it remains only to

(i) call on Sublemma 6 of Section 7 to obtain $p\{T_{[\varepsilon' n]} > n\} < D_3\theta_3^n$ for some

$\varepsilon' > 0$ (the proof is essentially the same as before); and to

(ii) verify once again that

$$p\left\{R > T_{[\varepsilon' n]} + n_0\right\} \le (1-\varepsilon_1)^{\frac{\varepsilon' n}{n_0}}.$$

9. Logistic Maps

9.1. Results and discussions.

Let $f : [-1,1] \circlearrowleft$ be a C^2 unimodal map with $f'(0) = 0$ and $f''(0) \neq 0$. We assume *there exist* $\alpha, \delta > 0$ *and* $\lambda > 1, M \in \mathbb{Z}^+$ *s.t.*

(H1) *outside of* $(-\delta, \delta), f$ *is expanding in the sense that*
(i) *if* $x, \dots, f^{M-1}x \notin (-\delta, \delta)$, *then* $|(f^M)'x| \ge \lambda^M$;
(ii) *for any* k, *if* $x, \dots, f^{k-1}x \notin (-\delta, \delta)$ *and* $f^k x \in (-\delta, \delta)$, *then* $|(f^k)'x| \ge \lambda^k$;

(H2) *the critical orbit satisfies*
(i) $|f^n(0)| \ge e^{-\alpha n}$ $\forall n \ge 1$
(ii) $|(f^n)'(f0)| \ge \lambda^n$ $\forall n \ge 1$.

Theorem 7. *Let* f *be as above. Then*
(a) f *fits the model of Part I with an exponential estimate for* $\{R > n\}$;
(b) f *admits an absolutely continuous invariant probability measure* ν;
(c) *if* (f^n, ν) *is ergodic* $\forall n \ge 1$, *then* (f, ν) *has exponential decay of correlations and CLT.*

Needless to say, the model in Part I has to be suitably interpreted for noninvertible maps. We comment briefly below on related results and on how this setting fits into the class of maps to which the methods of this paper may apply.

Remarks. *1.* (H1) and (H2) are satisfied on positive measure sets of parameters in generic 1-parameter families of unimodal maps passing through "Misiurewicz points" ([BC2], [TTY]). This includes in particular the family $f_a : x \mapsto 1 - ax^2$. For $\{f_a\}$ there is a positive measure set of a's near 2 for which (f_a^n, ν) is ergodic for all $n \ge 1$ ([Y2]).

2. Result (b) in Theorem 7 has been proved many times under similar conditions; see e.g. [CE], [BC1] (also [BY1]), and [NS]. Jakobson [J] had the first proof of the existence of acim's for a positive measure set of parameters for $\{f_a\}$. Assertion (c) is proved in [Y2]; for a similar result see [KN].

3. The "bad set"–"recovery" process alluded to in the introduction is most transparent in this 1-d setting: $x = 0$ is clearly the "bad point", and (H1) and (H2) guarantee that $\forall x \in (-\delta, \delta), \exists\ p(x) \approx \log \frac{1}{x}$ s.t. $\forall j < p(x), f^j x$ stays sufficiently close to $f^j 0$ that

$$* \qquad |(f^j)'(fx)| \approx |(f^j)'(f0)| \ge \lambda^j$$

$$* \qquad |(f^{p(x)})'x| \gtrsim \lambda^{p(x)/2}.$$

This simple lemma is proved in [BC1]. It says that when an orbit gets near the "bad point", a loss of hyperbolicity of order $\sim \delta$ is compensated for in $\sim \log \frac{1}{\delta}$ iterates. This is what we called "exponential recovery" in the introduction, and it translates into an exponentially decaying tail for R.

Observe the difference between the recovery process in, for example, Section 7 and here. In Section 7, condition (H4) guarantees that *statistically*, most points belong in a component that grows long exponentially fast, but for a small set of points this could take arbitrarily long. By contrast, what we have here is that when an orbit gets near 0, its derivative is guaranteed to recover within a time period determined by its distance to 0. Condition (H2) above, however, is not robust under perturbations. It holds only for a positive measure – but nowhere dense – set of parameter values. In some sense, the statistical part in Section 7 corresponds to parameter selection here.

9.2. Construction of $f^R : \Lambda \circlearrowleft$.

The basic properties of maps satisfying (H1) and (H2) are developed in [BC1] and [BC2]. For an exposition of the parts relevant to us, see also [Y2].

For purposes of dividing $[-1,1]$ into intervals on which f' is comparable, we partition $(-\delta,\delta)$ as follows: for $k \geq$ some k_0, let $I_k = (e^{-(k+1)}, e^{-k}), I_{-k} = -I_k$, and partition each I_k into k^2 subintervals called $\{I_{kj}\}$ of equal length. We will sometimes refer to $[-1,-\delta)$ and $(\delta,1]$ as I_{kj}-intervals. The following are proved in [BC1]:

(i) we may think of the function $p(\cdot)$ in 9.1 as constant on $I_{k,j}$-intervals; and for $x \in (-\delta,\delta)$, if $f^i x \in I_{k,j}$ for some $i \leq p(x)$, then $|f^i[0,x]| \ll |I_{k,j}|$;

(ii) there exists $C > 0$ such that for all n and all intervals ω with $f^i\omega$ lying in 3 adjacent I_{kj}'s for all $i \leq n$, we have that $(f^i)'x/(f^i)'y \leq C$ for all $x, y \in \omega$.

Let ω be as above. We may think of $f^i\omega, i = 0, 1, \cdots, n$, as being in one of two phases: suppose n_1 is the first time ω enters $(-\delta,\delta)$ and $p_1 = p(x)$ for $x \in f^{n_1}\omega$, then following the language of [BC1] we say that $\omega, f\omega, \cdots, f^{n_1}\omega$ are "free" while $f^{n_1+1}\omega, \cdots, f^{n_1+p_1}\omega$ are in a "bound" state. After time n_1+p_1, $f^i\omega$ is "free" again until the next time it enters $(-\delta,\delta)$ when a second "bound" period will begin, and so on.

In this setting we may take Λ to be an interval. We will in fact first work with $\Lambda = \Lambda^+ \cup \Lambda^-$ where Λ^+ and Λ^- are 2 intervals contained respectively in $I_{\pm k_0}$. Later on we could eliminate one of them as in 7.6 if we so desire. Return times are defined using an auxiliary partition $\tilde{\mathcal{P}}_n$ as before. Let $\omega \in \tilde{\mathcal{P}}_{n-1}$. For $n \leq R_0, \tilde{\mathcal{P}}_n$ partitions ω according to the I_{kj}-locations of its f^n-images, and partition points are inserted only to ensure that the f^n-images of $\tilde{\mathcal{P}}_n$-elements do not intersect more than 3 I_{kj}'s. For $n > R_0$, do as above except that if $f^n\omega \supset 3\Lambda^+$ or $3\Lambda^-$ ($3\Lambda^\pm :=$ the interval centered at the midpoint of $\Lambda^\pm$ and 3 times its length) then set $R = n$ on $f^{-n}\Lambda^\pm$ and continue iterating $\omega - \{R \leq n\}$ as before.

This construction may at first sight seem problematic since $f^n\omega$ may intersect an infinite number of $I_{k,j}$'s. Recall, however, from 1.2 and the last paragraph in 3.5

that we are only required to have a finite number of elements of $\mathcal{D}$ on the ℓth level for $\ell \leq R_0$, and that R_0 depends only on f, not on Λ. The requirement above is easily met by choosing $\Lambda^\pm$ so that $f^i\Lambda^\pm \cap \{0\} = \emptyset \ \forall i \leq R_0$.

We have now arranged for (P2) except for the a.e. finiteness of R; (P1) is trivial, and (P3) and (P5) are void in this case. To verify (P4), observe first that in our construction above, ω is partitioned only if $f^n\omega$ is "long", which, by observation (i) can happen only when $f^n\omega$ is in a "free" state. That is to say, separation can occur only when the points in question are free. Moreover, by (H1) and Remark 3 in 9.1, if an orbit beginning at z is in a free state at time k, then $(f^k)'z \geq \lambda^{\frac{k}{2}}$. This applied to $z = f^n x$ and $f^n y$ and $k = s(x,y) - n$ gives (P4)(a).

As for (P4)(b), observe that it is used solely for the purpose of proving Lemma 1 (3), which for noninvertible maps is a weaker assertion than (P4)(b) itself. Thus it suffices for us to prove Lemma 1 (3) directly, and that is a slight extension of (ii) above:

Sublemma 1. *There exists a constant C' such that if ω is as in (ii) and $f^n\omega \supset 3\Lambda^\pm$, then for all $x, y \in \omega$,*

$$\log \frac{(f^n)'x}{(f^n)'y} \leq C'|f^n x - f^n y|.$$

Proof. In the proof of (ii) it is shown that for $i \leq n$,

$$\log \frac{(f^i)'x}{(f^i)'y} \leq \text{ const. } \sum_{k \in S} \frac{|f^{i_k}x - f^{i_k}y|}{e^{-k}} \leq \text{ const. } \sum_k \frac{1}{k^2} := C$$

where k is positive, i_k is the last time when a bound period for $f^j\omega$, $j \leq i$, is initiated from $I_{\pm k}$, and S is the set of relevant k's. Applying this inequality to points in $f^{i_k}\omega$ for the time intervals $[i_k, n]$, we obtain that

$$\frac{|f^{i_k}x - f^{i_k}y|}{|f^{i_k}\omega|} \leq C\frac{|f^n x - f^n y|}{|f^n\omega|}.$$

Plugging this back in the first inequality, we conclude that

$$\begin{aligned}
\log \frac{(f^n)'x}{(f^n)'y} &\leq \text{ const. } \sum_{k \in S} \frac{|f^{i_k}\omega|}{e^{-k}} \cdot \frac{|f^{i_k}x - f^{i_k}y|}{|f^{i_k}\omega|} \\
&\leq \text{ const. } \left(\sum_{k \in S} \frac{|f^{i_k}\omega|}{e^{-k}} \right) C \frac{|f^n x - f^n y|}{|\Lambda^\pm|} \\
&\leq C'|f^n x - f^n y|.
\end{aligned}$$

■

9.3. Tail estimate for R.

To finish we need to argue that $\mu\{x \in \Lambda^{\pm} : R(x) > n\} \le C\theta^n$ for some $\theta < 1$. The picture here is considerably simpler since $\Omega = \Omega_n = \Lambda$. Consider $\omega \in f^k\tilde{\mathcal{P}}_k$, and let $\tilde{\mathcal{P}}_n^{\omega} := (f^k\tilde{\mathcal{P}}_{n+k}) \mid \omega$ as before. We define for $x \in \omega$

$$T(x) = \text{ the smallest } n \text{ s.t. } f^n(\tilde{\mathcal{P}}^{\omega}_{n-1}(x)) \text{ crosses } 3\Lambda^+ \text{ or } 3\Lambda^-.$$

The following lemma is proved in [BC2] (see [BY2] for more details):

Lemma. [BC2]. $\exists D_1' > 0$ and $\theta_1' < 1$ independent of ω s.t. if $\omega \approx I_{kj}$ then $\forall n \ge 6|k|$,

$$\mu\{x \in \omega : T(x) > n\} \le D_1'\theta_1'^n|\omega|.$$

As in previous sections, when $\omega \in \tilde{\mathcal{P}}_{k-1}$ makes a regular return to $\Lambda^{\pm}$ at time k and $\{\omega_i\} = f^k(\tilde{\mathcal{P}}_k \mid (\omega - \{R = k\}))$, we need to have a version of the lemma above for $\cup\omega_i$ with T defined individually on each ω_i. Here the argument is simple because there are no gaps in $\Lambda^{\pm}$, and $\mu(\bigcup_i\{x \in \omega_i : T(x) > n\})$ is easily estimated to be

$$\begin{aligned}
&\le\ 2e^{-\frac{n}{6}}\ +\sum_{\substack{i\ :\ \omega_i \approx \text{ some } I_{kj} \\ \text{with } |k| \le n/6}} \mu\{x \in \omega_i : T(x) > n\} \\
&\le\ 2e^{-\frac{n}{6}} + D_1'\theta_1'^n\ \ \le D_1\theta_1^n
\end{aligned}$$

for some $D_1 > 0$ and $\theta_1 < 1$. An argument similar to that in Step 3 of Section 7 finishes the proof.

10. Hénon-type Attractors

10.1 Statement of results.

Let $T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T_{a,b}(x,y) = (1 - ax^2 + y,\ bx).$$

It is an elementary fact that for $a < 2$ and $b > 0, b$ sufficiently small depending on $a, f = T_{a,b}$ has an attractor Σ in the sense that there is an open set $U \supset \Sigma$ with the property that $\forall z \in U, f^n z \to \Sigma$ as $n \to \infty$. (This is also true for $b < 0$ except that U is not known to be a neighborhood of Σ.) In [BC2] Benedicks and Carleson developed a machinery for analyzing the dynamics of $T_{a,b}$ on Σ for (a,b) in a positive measure set Δ near $a = 2$ and $b = 0$. Building on their analysis we prove the following result:

Theorem 8. (joint with M. Benedicks [BY3]). *Let* $f = T_{a,b}, (a, b) \in \Delta$. *Then*
(a) f *fits the model of Part I with an exponential estimate for* $\{R > n\}$*;*
(b) f *admits an SRB measure* ν*; and*
(c) (f, ν) *has exponential decay of correlations and CLT.*

The existence of an SRB measure is proved in [BY2]; part (b) above gives an alternate proof independent of this earlier work. In [BY2] it is also proved that (f^n, ν) is ergodic for all $n \geq 1$; this part is used in our proof of (c).

Remark. While we have stated our results for the Hénon family, none of the arguments depend on the specific formulas defining the Hénon maps. We do not attempt to give a complete characterization of the 1-parameter families to which [BC2] and our results apply, but point out only that the following properties play important roles: (a) the maps in question are strongly hyperbolic on large parts of phase space, and hyperbolicity is spoiled only by tangencies of stable and unstable manifolds; (b) these tangencies are quadratic, giving rise to unstable manifolds that are parabolas in shape; indeed, for the "good" parameters local unstable manifolds are either parabolas or "straight"; (c) if by varying parameters one is able to effectively change the locations of these tangencies and their images, then it is likely that the "good" parameters from a positive measure set. Let us loosely refer to attractors with these properties as *Hénon-type attractors.* For example, Hénon-type attractors have been shown to appear in certain homoclinic bifurcations (see e.g. [MV]) and our results apply to them as well.

A complete proof of Theorem 8 is given in [BY3].

10.2. A synopsis of the proof.

When b is small, $f = T_{a,b}$ is not far from the 1-d map $x \mapsto 1 - ax^2$, and the attractor Σ lies in a small neighborhood of $[-1, 1] \times \{0\}$ in $\mathbb{R}^2$. With the 1-d picture in mind, one sees immediately that away from the origin roughly horizontal vectors are expanded, and when an orbit passes near the origin these vectors could be rotated into all possible directions. Previously established expanding behavior may or may not be jeopardized depending on the position of this rotated vector as well as the future orbit of the point in question. While it is clear that the switching of stable and unstable directions is the main source of "badness", it is not clear *a priori* that the Hénon maps have well defined, localized "bad sets".

In [BC2] the authors introduced into the subject fairly general techniques for systematically identifying "bad sets". In the case of the Hénon maps they gave a complete analysis of the loss of hyperbolicity and subsequent recovery behavior. This is done via an inductive procedure which is successfully carried through only for parameters in Δ. We summarize in the next paragraph the dynamical picture for $f = T_{a,b},\ (a, b) \in \Delta$.

The attractor Σ is the closure of the unstable manifold W of a fixed point, and the "bad set" could be regarded as the closure of a set $\mathcal{C} \subset W$ called the "critical set" in [BC2]. $\mathcal{C}$ is located near the origin, and all $z \in \mathcal{C}$ lie on roughly horizontal pieces of W. Dynamically, a distinguishing feature of $z \in \mathcal{C}$ is that the tangent vector to W at

z is the asymptotically contracted direction under both Df_z^{-n} and Df_z^n as $n \to \infty$; in other words, points in $\mathcal{C}$ are homoclinic tangencies. Geometrically, $\mathcal{C}$ consists of the points of origin of all the "turns". Because temporary stable and unstable manifolds (i.e. those that work for finite time) have quadratic tangencies near $z \in \mathcal{C}$, the dynamics of $f|W$ near the turns resemble those of the maps $x \mapsto 1 - ax^2$. The idea of this last statement is used heavily in [BC2]; for a more transparent geometric picture see [BY2].

Let us assume now the picture in the last paragraph and proceed to study $f^R : \Lambda \circlearrowleft$. We will not delve into technical details of the Hénon maps here, but will compare and contrast its various aspects with the other examples. For our purposes the Hénon picture is a hybrid of those in Sections 7 and 9, although somewhat more complicated. As in the case of the logistic maps, it suffices to construct Λ out of 2 boxes $Q^\pm$ located to the left and right of $\mathcal{C}$, with $Q^\pm$ having the full thickness of the attractor in the vertical direction; this is because roughly horizontal segments of unstable leaves cannot grow indefinitely without cutting across the critical set and hence Q^+ or Q^-. As in the case of hyperbolic maps with discontinuities, our hyperbolic product sets $\Lambda^\pm \subset Q^\pm$ are necessarily products of Cantor sets because stable and unstable manifolds do not form product structures on open subsets of Σ. Following earlier strategy, we take $\Lambda^\pm = \{z \in Q^\pm : \text{dist}(f^n z, \mathcal{C}) > \delta e^{-\alpha n} \; \forall n \geq 0\}$, although $\text{dist}(\cdot, \mathcal{C})$ here needs to be defined with greater care due to the fractal nature of $\mathcal{C}$. Finally, what is responsible for the exponential estimate in $\mu_\gamma\{R > n\}$ is the fact that properly controlled segments of W that pass near $\mathcal{C}$ are guaranteed to grow back to "full length" at exponential speeds. The recovery mechanism here is similar to that for the logistic maps. It is a consequence of the fact that the "turns" are only allowed to approach the critical set very slowly. As in the 1-d case, this is a special property of the parameters selected.

References

[BC1] M. Benedicks and L. Carleson, *On iterations of* $1-ax^2$ *on* $(-1,1)$, Annals of Math., **122** (1985), 1-25.

[BC2] M. Benedicks and L. Carleson, *The dynamics of the Henon map*, Ann. Math., **133** (1991), 73-169.

[BY1] M. Benedicks and L.-S. Young, *Absolutely continuous invariant measures and random perturbations for certain one-dimensional maps*, Ergod. Th. & Dynam. Sys., **12** (1992), 13-37.

[BY2] M. Benedicks and L.-S. Young, SBR *measures for certain Henon maps*, Inventiones Math., **112** (1993), 541-576.

[BY3] M. Benedicks and L.-S. Young, *Decay or correlations for certain Henon maps*, 1996 preprint

[B] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer Lecture Notes in Math., **470** (1975).

[BSC1] L. A. Bunimovich, Ya. G. Sinai and N. I. Chernov, *Markov partitions for two-dimensional billiards*, Russ. Math. Surv., **45** (1990), 105-152.

[BSC2] L. A. Bunimovich, Ya. G. Sinai and N. I. Chernov, *Statistical properties of 2-dimensional hyperbolic billiards*, Russ. Math. Surv., **46** (1991), 47-106.

[C1] N. I. Chernov, *Ergodic and statistical properties of piecewise linear hyperbolic automorphism of the 2-torus*, J of Stat. Phys., **69** (1992), 111-134.

[C2] N. I. Chernov, *Limit theorems and Markov approximations for chatoic dynamical systems*, Probab. Theory Relat. Fields, **101** (1995), 321-362.

[CELS] N. I. Chernov, G. Eyink, J. Lebowitz and Ya. G. Sinai, *Steady-state electrical conduction in the periodic Lorentz gas*, Commun. Math. Phys. **154** (1993) 569-601.

[CE] P. Collet and J. P. Eckmann, *Positive Lyapunov exponents and absolutely continuity*, Ergod. Th. & Dynam Syst., **3** (1983), 13-46.

[CL] P. Collet and Y. Levy, *Ergodic properties of the Lozi mappings*, Commun. Math. Phys., **93** (1984), 461-482.

[D&S] Dunford and Schwartz, *Linear operators*, Part I: General Theory, 1957.

[G] M. I. Gordin, *The central limit theorem for stationary processes*, Soviet Meth. Dokl., **10(5)** (1969), 1174-1176

[HK] F. Hofbauer and G. Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Math. Z., **180** (1982), 119-140.

[H] H. Hu, *Conditions for the existence of SRB measures for "almost Anosov" diffeomorphisms*, 1995 preprint

[HY] H. Hu and L.-S. Young, *Nonexistence of* SBR *measures for some systems that are "almost Anosov"*, Erg. Th. & Dyn. Sys., **15** (1995), 67-76.

[I] S. Isola, *Dynamical zeta functions and correlation functions for intermittent interval maps*, preprint.

[J] M. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Commun. Math. Phys., **81** (1981), 39-88.

[Ka] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, Publ. Math. IHES, **51** (1980), 137-174.

[KS] A. Katok and J. M. Strelcyn, *Invariant manifolds, entropy and billiards; smooth maps with singularities*, Springer Lecture Notes in Math. **1222** (1986).

[KV] C. Kipnis and S.R.S. Varadhan, *Central limit theorem for additive functions of reversible Markov process and applications to simple exclusions*, Commun. Math. Phys. **104** (1986), 1-19.

[Ke] G. Keller, *Un théorem de la limite centrale pour une classe de transformations monotones par morceaux*, Comptes Rendus de l'Académie des Sciences, Série A **291** (1980), 155-158.

[KN] G. Keller and T. Nowicki, *Special theory, zeta functions and the distributions of periodic points for Collet-Eckmann maps*, Commun. Math. Phys., **149** (1992), 31-69.

[LaY] A. Lasota and J. Yorke, *On the existence of invariant measures for piecewise monotonic transformations*, Trans. Am. Math. Soc., **186** (1973), 481-488.

[LY] F. Ledrappier and L.-S. Young, *The metric entropy of diffeomorphisms*, Annals of Math., **122** (1985), 509-574.

[L1] C. Liverani, *Decay of correlations*, Annals Math. **142** (1995), 239-301.

[L2] C. Liverani, *Central limit theorem for deterministic systems*, International conference on dynamical systems, Montevideo 1995, Eds. F.Ledrappier, J.Lewowicz, S.Newhouse, Pitman research notes in Math, **362** 56-75, (1996).

[MV] L. Mora and M. Viana, *Abundance of strange attractors*, Acta Math. **171** (1993), 1-71.

[NS] T. Nowicki and S. van Strein, *Absolutely continuous invariant measures under a summability condition*, Invent. Math., **105** (1991), 123-136.

[O] V. I. Oseledec, *A multiplicative ergodic theorem: Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc., **19** (1968), 197-231.

[PP] W. Parry and M. Pollicott, *Zeta functions and the period structure of hyperbolic dynamics*, Société Mathématique de France (Astérisque Vol. no **187-188**), Paris, 1990.

[P1] Ya. B. Pesin, *Characteristic Lyapunov exponents and smooth ergodic theory*, Russ. Math.

Surveys, **32** (1977), 55=114.

[P2] Ya. B. Pesin, *Dynamical systems with generalized hyperbolic attractors: Hyperbolic, ergodic and topological properties*, Ergodic Theory Dyn. Syst., **12** (1992), 123-151.

[PS] C. Pugh and M. Shub, *Ergodicity of Anosov actions*, Inventiones Math., **15** (1972), 1-23.

[R1] D. Ruelle, *A measure associated with Axiom A attractors*, Amer. J. Math., **98** (1976), 619-654.

[R2] D. Ruelle, *Thermodynamic formalism*, Addison-Wesley, New York, 1978.

[R3] D. Ruelle, *Flots qui ne mélangent pas exponentiellement*, C. R. Acad. Sci. Paris Sér. I Math **296** (1983), 191-193.

[Ry] M. Rychlik, *Invariant measures and the variational principle for Lozi mappings*, Ph.D. Thesis (unpublished), Berkeley 1983.

[S1] Ya. G. Sinai, *Gibbs measures in ergodic theory*, Russ. Math. Surveys, **27(4)** (1972), 21-69.

[S2] Ya. G. Sinai, *Dynamical systems with elastic relections: Ergodic properties of dispersing billiards*, Russ. Math. Surveys, **25(2)** (1970), 137-189.

[TTY] P. Thieullen, C. Tresser and L.-S. Young, *Positive Lyapunov exponent for generic* 1-*parameter families of unimodal maps*, C.R. Acad. Sci. Paris, t. 315 Série I (1992), 69-72; J'Analyse, **64** (1994), 121-172.

[W] M. Wojtkowski, *Invariant families of cones and Lyapunov exponents*, Erg. Th. and Dyn. Sys., **5** (1985), 145-161.

[Y1] L.-S. Young, *A Bowen-Ruelle measure for certain piecewise hyperbolic maps*, Trans. Am. Math. Soc., **287** (1985), 41-48.

[Y2] L.-S. Young, *Decay of correlations for certain quadratic maps*, Commun. Math. Phys., **146** (1992), 123-138.