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REVIEW

Mathematical theory of Lyapunov exponents

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Abstract
This paper reviews some basic mathematical results on Lyapunov exponents, one of the most fundamental concepts in dynamical systems. The first few sections contain some very general results in nonuniform hyperbolic theory. We consider (f, μ), where f is an arbitrary dynamical system and μ is an arbitrary invariant measure, and discuss relations between Lyapunov exponents and several dynamical quantities of interest, including entropy, fractal dimension and rates of escape. The second half of this review focuses on *observable chaos*, characterized by positive Lyapunov exponents on positive Lebesgue measure sets. Much attention is given to SRB measures, a very special kind of invariant measures that offer a way to understand observable chaos in dissipative systems. Paradoxical as it may seem, given a concrete system, it is generally impossible to determine with mathematical certainty if it has observable chaos unless strong geometric conditions are satisfied; case studies will be discussed. The final section is on noisy or stochastically perturbed systems, for which we present a dynamical picture simpler than that for purely deterministic systems. In this short review, we have elected to limit ourselves to finite-dimensional systems and to discrete time. The phase space, which is assumed to be $\mathbb{R}^d$ or a Riemannian manifold, is denoted by M throughout. The Lebesgue or the Riemannian measure on M is denoted by $\mathfrak{m}$, and the dynamics are generated by iterating a self-map of M, written $f : M \circlearrowleft$. For flows, the reviewed results are applicable to time-t maps and Poincaré return maps to cross-sections.

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1. Nonuniformly hyperbolic systems

We begin with a quick review of the setting of nonuniform hyperbolic theory, mostly to fix notation but we will also take the opportunity to bring up some issues.

Given a differentiable map $f : M \circlearrowleft$, a point $x \in M$ and a tangent vector v at x, we define

$$\lambda(x, v) = \lim_{n\to\infty} \frac{1}{n} \log |Df_x^n(v)| \qquad \text{if this limit exists;} \tag{1}$$

and if it does not, then the limit is replaced by lim inf or lim sup, and we write $\underline{\lambda}(x, v)$ and $\overline{\lambda}(x, v)$, respectively. Thus, $\lambda(x, v) > 0$ means that $|Df_x^n(v)|$ grows exponentially, and that is interpreted to mean exponential divergence of nearby orbits. Such an interpretation is valid for as long as the orbits in question remain very close to one another; once they move apart, $\lambda(x, v)$ offers no information.

While limits of the type in (1) need not exist at every x, they do exist almost everywhere under stationarity assumptions: the multiplicative ergodic theorem [O] tells us that given an f-invariant Borel probability measure μ, the following hold at μ-a.e. x: there are numbers

$$\lambda_1(x) > \lambda_2(x) > \ldots > \lambda_{r(x)}(x)$$

with multiplicities $m_1(x), \ldots, m_{r(x)}(x)$, respectively, such that

(i) for every tangent vector v at x, $\lambda(x, v) = \lambda_i(x)$ for some i,
(ii) $\sum_i m_i(x) = \dim(M)$ and
(iii) $\sum_i \lambda_i(x)m_i(x) = \lim_{n\to\infty} \frac{1}{n} \log |\det(Df_x^n)|$.

If f is a diffeomorphism, i.e. if it is invertible, then there is a decomposition of the tangent space T_xM into

$$T_xM = E_1(x) \oplus \ldots \oplus E_{r(x)}(x),$$

where $\dim E_i(x) = m_i(x)$ and $\lambda(x, v) = \lambda_i(x)$ for $v \in E_i(x)$. The numbers $\{\lambda_i, m_i\}$ are called the *Lyapunov exponents* of the system (f, μ). If (f, μ) is ergodic, then λ_i and m_i are constant μ-a.e.

Given a differentiable map f and an f-invariant Borel probability measure μ, many general facts about the system (f, μ) have been proved. The most basic of these facts translates the *infinitesimal* information given by Lyapunov exponents to *local* information along orbits for the nonlinear map f. In the conservative case, i.e. where μ is equivalent to $\mathfrak{m}$, the results in the next paragraph were first proved by Pesin [Pe]; in the generality discussed here, they are due to Ruelle [R2].

Let us assume for definiteness that f is a C^2 (or $C^{1+\alpha}$, $\alpha > 0$) diffeomorphism of M. Then, at μ-a.e. x for which

$$E^u(x) := \oplus_{i:\lambda_i(x)>0} E_i(x) \qquad \text{and} \qquad E^s(x) := \oplus_{i:\lambda_i(x)<0} E_i(x)$$

are nontrivial, there is a local stable disc $W^s_{\text{loc}}(x)$ and a local unstable disc $W^u_{\text{loc}}(x)$ tangent to $E^s(x)$ and $E^u(x)$, respectively. These discs are defined μ-a.e. and are invariant, meaning $f(W^s_{\text{loc}}(x)) \subset W^s_{\text{loc}}(fx)$ and $f^{-1}(W^u_{\text{loc}}(x)) \subset W^u_{\text{loc}}(f^{-1}x)$. The sizes and directions of $W^u_{\text{loc}}(x)$ and $W^s_{\text{loc}}(x)$ vary measurably with x; so outside a small positive μ-measure set, they vary continuously, the two families of discs forming a kind of local coordinate system for f. The global unstable manifold at x, denoted $W^u(x)$, is defined as

$$W^u(x) := \left\{ y \in M : \liminf_{n\to\infty} \frac{1}{n} \log d(f^{-n}x, f^{-n}y) < 0 \right\}$$

and is equal to $\cup_{n\geqslant 0} f^n(W^u_{\text{loc}}(f^{-n}x))$. Global stable manifolds are defined similarly.

A number of other results have been proved for (f, μ) under the assumption that some or all of λ_i are nonzero; some of them are reviewed in the next sections. I think it is fair to say that this theory, often referred to as *nonuniform hyperbolic theory*, has worked out quite well; the class of dynamical systems to which it applies is considerably broader than Axiom A systems [Sm] or the systems studied by Anosov [An].

One of the caveats when trying to apply nonuniform hyperbolic theory to concrete systems is that the properties of (f, μ) depend not only on f but also on μ. In general, systems that are more complicated than, say, gradient flows admit many ergodic measures. Since distinct ergodic measures are mutually singular, they can offer quite different statistical descriptions of the same map. Which invariant measures, then, are more 'representative'? The answer will depend on one's goals. Sometimes there are natural choices, such as the Liouville measure for Hamiltonian systems. At other times, identifying a suitable μ can be a problem in itself. We will return to this question in section 4 with specific issues in mind.

2. Lyapunov exponents and entropy

2.1. Entropy as a measure of dynamical complexity

Following Kolmogorov and Sinai, the entropy of a measure-preserving transformation T of a probability space $(X, \mathcal{B}, \mu)$ is defined as follows. Let $\alpha = \{A_1, A_2, \ldots, A_k\}$ be a measurable partition of X. For $n < m$, write $\alpha_n^m = T^{-n}\alpha \vee \cdots \vee T^{-m}\alpha$, where $\alpha \vee \beta = \{A \cap B, A \in \alpha, B \in \beta\}$ is the join of the two partitions. Letting

$$H(\alpha) = -\sum_i p_i \log p_i, \qquad \text{where} \qquad p_i = \mu(A_i),$$

we define the *entropy* of T to be

$$h_\mu(T) = \sup_\alpha h(T; \alpha), \qquad \text{where} \qquad h(T; \alpha) = \lim_{n\to\infty} \frac{1}{n} H\big(\alpha_0^{n-1}\big).$$

Here, $H(\alpha)$ has the interpretation of the expected information gain or the amount of uncertainty removed, upon learning the α-address of a randomly chosen point. Let $\alpha(x)$ denote the element of α containing x. Then, $\alpha_0^{n-1}(x) = \{y \in X : \alpha(T^i x) = \alpha(T^i y) \text{ for } 0 \leqslant i < n\}$, i.e. each element of α_0^{n-1} represents a distinguishable n-itinerary. Thus, $h(T; \alpha)$ is the per-iterate information gain upon learning an n-itinerary as $n \to \infty$. A good mathematical text for entropy is [Wa].

The Shannon–McMillan–Breiman theorem offers another interpretation. It states that, under the conditions above and assuming additionally that (T, μ) is ergodic,

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu\alpha_0^{n-1}(x) = h(T, \alpha) \qquad \text{a.s.}$$

In other words, if $h = h(T, \alpha)$, then the following holds. Given any $\varepsilon > 0$, there exists N such that for all $n \geqslant N$, there is a set $X_n \subset X$ with $\mu(X_n) > 1 - \varepsilon$ such that X_n consists of $\sim e^{n(h\pm\varepsilon)}$ elements of α_0^{n-1} each having measure $\sim e^{-n(h\pm\varepsilon)}$. Viewing the elements of α_0^{n-1} in X_n as representing 'typical' n-itineraries, the SMB theorem states that a system has entropy h if the number of 'typical' n-itineraries grows like $\sim e^{nh}$. This gives an intuitive meaning for entropy.

Unlike Lyapunov exponents, which measure local instability in terms of geometric distances between orbits, entropy is a purely probabilistic way to quantify dynamical complexity. The fact that there is little *a priori* reason to expect these two sets of dynamical invariants to be related is a testimony to the depth of the following results.

2.2. Relation between entropy and Lyapunov exponents

The results below hold for arbitrary C^2 diffeomorphisms $f : M \circlearrowleft$ and arbitrary f-invariant Borel probability measures which we assume to be compactly supported. These results are very general; no other conditions are needed except for what is explicitly stated. As before, let $\lambda_1 > \ldots > \lambda_r$ denote the distinct Lyapunov exponents of (f, μ), and let E_i be the subspaces corresponding to λ_i. Also, let $a^+ := \max\{a, 0\}$.

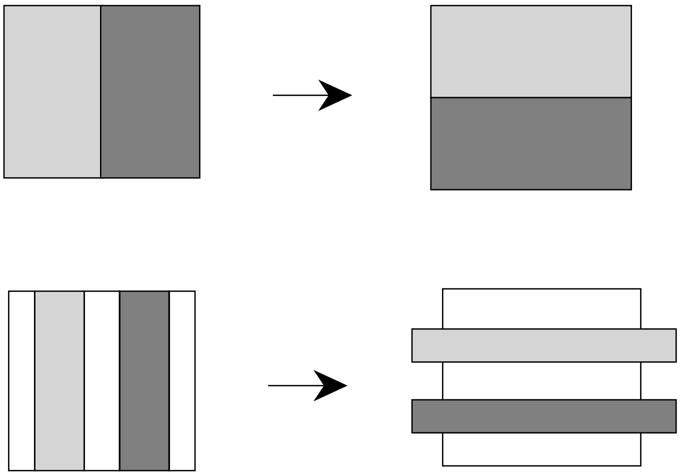

Figure 1. Relation between entropy and Lyapunov exponents.

Theorem 1. *Let (f, μ) be as above. Then*

(i) [R3] In general,

$$h_\mu(f) \leqslant \int \sum_i \lambda_i^+ \dim(E_i) \, \mathrm{d}\mu.$$

(ii) [Pe] If μ is equivalent to $\mathfrak{m}$ (=Lebesgue measure), then

$$h_\mu(f) = \int \sum_i \lambda_i^+ \dim(E_i) \, \mathrm{d}\mu.$$

(iii) [LY1] When $\lambda_1 > 0$, the equality in (ii) holds if and only if μ is an SRB measure.

We will return to the meaning of SRB measures later. Here is the formal definition: an f-invariant Borel probability measure μ is called an *SRB measure* if (a) $\lambda_1 > 0$ μ-a.e. (so that unstable manifolds are defined μ-a.e.) and (b) the conditional probabilities of μ on unstable manifolds are 'smooth'. For a measure on M, 'smooth' means having a density with respect to $\mathfrak{m}$. SRB measures need not be smooth themselves, but (b) says their conditional measures on unstable manifolds are smooth with respect to the Riemannian measure on these manifolds.

The results in theorem 1 have the following interpretation: (i) states that all uncertainty in the prediction of future events comes from positive Lyapunov exponents, though not all expansion will necessarily result in uncertainty, i.e. there can be 'wasted expansion'. (ii) states that there is no wasted expansion in conservative systems, that is to say, in conservative systems, all expansion goes into the creation of entropy. (iii) is a clarification of (ii); it explains that with regard to the prediction of future events, it is only what happens in the unstable direction that counts; what happens in stable directions is irrelevant. Note that (iii) also implies that invariant measures that are equivalent to Lebesgue measures are SRB measures when they have positive Lyapunov exponents.

These ideas are illustrated in the two examples in figure 1. Without a doubt, these examples are overly simplistic, but I think they capture what is going on. In each example, the map stretches and contracts each of the shaded regions linearly and maps them onto the regions on the right with the same shading. For the map on top, (f, μ), where μ is the Lebesgue measure, is isomorphic to the $\left(\frac{1}{2}, \frac{1}{2}\right)$-Bernoulli shift, and $h_\mu(T) = \lambda_1 = \log 2$, illustrating item (ii) in theorem 1. The second map also admits an invariant measure μ with the property that (f, μ) is isomorphic to the $\left(\frac{1}{2}, \frac{1}{2}\right)$-Bernoulli shift. Here, μ is singular with respect to $\mathfrak{m}$,

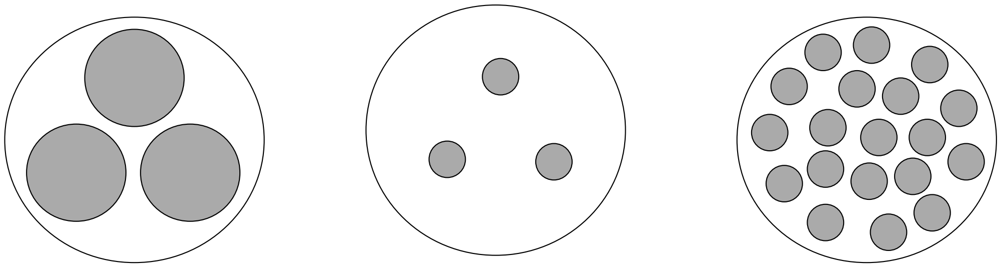

Figure 2. Entropy, Lyapunov exponents and fractal dimension.

and $\lambda_1 > \log 2 = h_\mu(T)$, showing that the inequality in (i) can be strict. For this map, the expansion is stronger than needed, in the sense that the parts that 'spill over' the sides of the box do not contribute to entropy; this is what we meant by 'wasted expansion'. Re-examining the two maps, we see that whether or not $\lambda_1 = \log 2$ has to do only with what happens in the expanding direction: the equality holds if and only if μ is smooth in the horizontal direction; this is the gist of item (iii) in theorem 1.

For further elucidation of the results in theorem 1, see section 3.1, where refinements of these results are explained in a bit more detail.

3. Between entropy and sum of positive exponents

Interpretations of the gap in the entropy inequality in theorem 1(i) are discussed in this section. In section 3.1, it is connected in a precise way to the fractal dimension of the invariant measure, and in section 3.2, it is connected to rates of escape from neighborhoods of (non-attracting) invariant sets.

3.1. Fractal dimension

The setting and notation are as in section 2.2. For simplicity, we assume here that (f, μ) is ergodic, so its Lyapunov exponents are given by a finite set of numbers $\lambda_1 > \ldots > \lambda_r$ with multiplicities $m_i = \dim(E_i)$.

To understand the relation between the entropy and the sum of positive Lyapunov exponents, it is simplest to first consider a situation where in the regions of interest the map is a uniform dilation. Figure 2 depicts three mappings of this type. In each case, a finite number of smaller circular discs B_i lie within a larger disc B. Each B_i is mapped affinely onto B, and we assume $f^{-1}(B) = \cup_i B_i$. The set of interest is $\Lambda = \cap_{n\geqslant 0} f^{-n}B$. Here, the entropy h is equal to $\log k$, where k is the number of pre-images, i.e. the number of B_i, and the Lyapunov exponent λ, which has multiplicity 2, is the logarithm of the dilation factor. Comparing (a) and (b) in figure 2 , it is clear that with h being fixed, the fractal dimension of Λ decreases as λ increases, and comparing (b) and (c), we see that with λ being fixed, the fractal dimension goes up with entropy. Indeed, the computation here is exact, and it gives $\dim(\Lambda) = h/\lambda$.

With some technical work, the idea of this example can be turned into a mathematical result, which we state as theorem 2′ (it is a special case of theorem 2). The setting is as above, except that we allow f to be noninvertible.

Theorem 2′. *If $\lambda_i \equiv \lambda > 0$ for all i, then $h_\mu(f) = \lambda \cdot \dim(\mu)$.*

Here, $\dim(\mu)$, the *dimension* of μ, is defined as follows. It is equal to the number α (if such a number exists) with the property that for μ-a.e. x, $\mu B(x, \varepsilon) \sim \varepsilon^\alpha$ as $\varepsilon \to 0$, where

$B(x, \varepsilon)$ is the ball of radius ε centered at x. For example, if μ is the d-dimensional Lebesgue measure, then $\alpha = d$. To see why the result in theorem 2′ is true, consider

$$B(x, \varepsilon; n) = \{y \in M : d(f^k x, f^k y) < \varepsilon \; \forall \, 0 \leqslant k < n\}.$$

Then, for a μ-typical point x, for small enough ε and large enough n, by the definition of Lyapunov exponents, we have

$$B(x, \varepsilon; n) \sim B(x, \varepsilon e^{-\lambda n}). \tag{2}$$

We claim that a variant of the Shannon–McMillan–Breiman theorem gives

$$\mu B(x, \varepsilon; n) \sim e^{-nh}, \tag{3}$$

where $h = h_\mu(f)$. This is because if the elements of a partition α are essentially ε-balls, then, for most x, $\alpha_0^{n-1}(x)$ is comparable to $B(x, \varepsilon, n)$. Comparing (2) and (3) and letting $\delta = e^{-n\lambda}$, we obtain, as $n \to \infty$,

$$\mu B(x, \delta) \sim \delta^{\frac{h}{\lambda}},$$

which implies $\dim(\mu) = \frac{h}{\lambda}$. This is the idea of the proof.

The situation in general is somewhat more complicated. The result can be summarized as follows.

Theorem 2. *[LY2] Let (f, μ) be as in the beginning of this subsection. Then, corresponding to each positive Lyapunov exponent λ_i, there is a number $\delta_i \in [0, m_i]$ such that if $\mu|W^u$ denotes the conditional measures of μ on W^u-leaves, then*

$$\dim(\mu|W^u) = \sum_{i:\lambda_i>0} \delta_i \qquad \text{and} \qquad h_\mu(f) = \sum_i \lambda_i^+ \delta_i.$$

The numbers δ_i can be interpreted as the 'partial dimensions' of μ in the directions of E_i; the first equality above states that they add up to the dimension of the conditional measures on W^u. These quantities can be defined precisely by (i) looking at hierarchies of unstable manifolds $W^1 \subset W^2 \subset \cdots \subset W^u$, where W^k corresponds to the largest k positive Lyapunov exponents and (ii) introducing a notion of entropy that measures randomness along W^k-manifolds while ignoring randomness in transverse directions. We will not go into further details, except to point out that the formula $h = \sum_i \lambda_i^+ \delta_i$ in theorem 2 is a refinement of the results in theorem 1: since $\delta_i \leqslant m_i$, it implies theorem 1(i), and if μ is smooth or is an SRB measure, then $\delta_i = m_i$, which gives the entropy equality in theorem 1.

3.2. Escape rates

The setting here consists of a triple $(f, M; H)$, where $f : M \circlearrowleft$ is as usual and $H \subset M$ is an open set, to be thought of as a 'hole' through which mass is allowed to escape. We follow the trajectories in M until they enter H; once a point enters H, it leaves the system forever, i.e. we stop considering it. Small holes are often used to model small (unintended) leaks in physical systems. Questions of escape from neighborhoods of non-attracting invariant sets can also be treated in this framework. Let Λ be such a set, and view $H = M \setminus \bar{U}$, where U is a neighborhood of Λ. Although they do not capture asymptotic dynamics as $t \to \infty$, non-attracting invariant sets can significantly influence the dynamical picture depending on how long orbits are 'stuck' near them, i.e. depending on the rate of escape from U.

Given $(f, M; H)$ and an initial distribution m on $M \setminus H$, the *escape rate* is defined to be $-\rho(m)$, where

$$\rho(m) = \lim_{n\to\infty} \frac{1}{n} \log m\big(\cap_{i=0}^n f^{-i}(M \setminus H)\big)$$

when the limit exists. We are primarily interested in the case where m has a density with respect to $\mathfrak{m}$ or is an SRB measure. Let $\Omega = \cup_{i\in\mathbb{Z}} f^{-i}(M \setminus H)$ be the largest invariant set which does not meet the hole, and let $\mathcal{I}(\Omega)$ denote the set of invariant Borel probability measures on Ω. As we will see, quantities of the form

$$P_\mu \ := \ h_\mu(f) - \int \lambda^+ \mathrm{d}\mu,$$

where $\mu \in \mathcal{I}(\Omega)$ and $\lambda^+ = \sum_i \lambda_i m_i$, are of relevance. The following is a prototypical result.

Theorem 3. *[Y3] Let $(f, M; H)$ be such that (i) Ω is compact with $d(\Omega, \partial H) > 0$, and (ii) $f|_\Omega$ is uniformly hyperbolic. Then, $\rho(\mathfrak{m})$ is well defined and satisfies*

$$\rho(\mathfrak{m}) \ = \ \sup\{P_\nu : \nu \in \mathcal{I}(\Omega)\};$$

in fact, $\rho(\mathfrak{m}) = P_\mu$ for some $\mu \in \mathcal{I}(\Omega)$.

We say $f|_\Omega$ is *uniformly hyperbolic* if there is a continuous splitting of the tangent space at every $x \in \Omega$ into $E^u \oplus E^s$, such that for some $\kappa > 1$, $|Df_x(v)| \geqslant \kappa|v|$ for all $v \in E^u$, and $|Df_x(v)| \leqslant \kappa^{-1}|v|$ for all $v \in E^s$. Under the above conditions, theorem 3 states that there is a variational principle and the escape rate is given by the maximum difference between the entropy and the sum of positive Lyapunov exponents counted with multiplicity. This suggests yet another interpretation of the gap in the entropy inequality in theorem 1(i): expansion pushes mass away from Ω, while the need to produce entropy keeps it from leaving—and the balance of the two gives the net escape rate. The escape rate from a neighborhood of a saddle fixed point, e.g., is given by the log of the unstable eigenvalue; here, Ω is the fixed point, and entropy on Ω is zero.

The ideas in the last paragraph are not universally valid (as mass can stay around without producing entropy), but they have been generalized to a large class of dynamical systems that exhibit a 'sufficient amount of hyperbolicity'. We refer the reader to [DWY] for details and for related references, and we mention here only one example of a map in this class, namely the 2D periodic Lorentz gas with small convex holes on the table. For this system, it has been proved that for large classes of initial distributions related to $\mathfrak{m}$, the escape rate is well defined and is given by the conclusions of theorem 3 with the definition of $\mathcal{I}(\Omega)$ slightly modified.

4. Observable chaos

In this section and the next, we adopt a viewpoint that equates observable events with positive Lebesgue measure sets and give importance to dynamical phenomena that are observable.

4.1. Positive exponents on positive Lebesgue measure sets

It is one thing for a dynamical system to have orbits that behave in chaotic ways, with $\lambda(x, v) > 0$ for some points x, another for this chaotic behavior is to be observable. In finite-dimensional dynamics, one often equates positive Lebesgue measure sets with observable events. Adopting such a view, let us say $f : M \circlearrowleft$ has *observable chaos* if $\lambda_{\max} > 0$ on at least a positive Lebesgue measure set, where

$$\lambda_{\max}(x) := \liminf_{n\to\infty} \frac{1}{n} \log \left\| Df_x^n \right\|,$$

i.e. $\lambda_{\max}$ is the largest Lyapunov exponent at x when that is defined. As we refer to this condition many times, let us abbreviate it as 'positive LE' so that in the rest of this review 'positive LE' has a precise meaning, namely $\lambda_{\max} > 0$ on a positive $\mathfrak{m}$-measure set.

Having a horseshoe implies the existence of orbits that behave chaotically. It does not imply positive LE, however, for the horseshoe itself occupies a zero Lebesgue measure set, and its presence does not preclude the possibility that orbits starting from $\mathfrak{m}$-a.e. $x \in M$ may tend eventually to a stable equilibrium (or a 'sink'). This, in fact, happens often, and systems with both 'horseshoes and sinks' are sometimes said to have *transient chaos*: orbits that start near a horseshoe may appear chaotic for a short time as they follow orbits within the horseshoe, but in time, almost all orbits tend to a stable equilibrium. By contrast, positive LE implies that the instability persists *for all future times* and occurs on a large enough set to be *observable*. It is a much stronger form of chaos than the presence of horseshoes alone.

For Hamiltonian (or volume preserving) systems, the meaning of positive LE is relatively straightforward: a system has positive LE if and only if with respect to the Liouville (or Lebesgue) measure, there is a positive Lyapunov exponent on at least a positive measure set. This is not to suggest that positive LE is easy to check in concrete situations, but at least we are clear on how it comes about.

For a dissipative system, the situation is more subtle: suppose orbits starting from an open set U tend toward an attractor Λ, which we assume is more complicated than a fixed point. Is it possible for such a system to have positive LE? The answer turns out to be yes, but the mechanism has to be different from that in the last paragraph, for the system here is not likely to have an invariant probability measure with a density. This is because (i) all invariant probability measures that live on U must in fact be supported on Λ, because the dynamics on $U \setminus \Lambda$ are transient, and (ii) if there is volume contraction—and there often is for an attractor to attract—then $\mathfrak{m}(\Lambda) = 0$.

The only known mechanism for a dissipative system with an attractor to have positive LE is via the idea of SRB measures, which we discuss next.

4.2. SRB measures

In the 1970s, there was a breakthrough in the ergodic theory of hyperbolic systems. The setting is that of a C^2 diffeomorphism f with a uniformly hyperbolic attractor Λ (see section 3.2 for the definition of uniform hyperbolicity). We assume that Λ is not a periodic sink, but permit it to be all of M (to include the case of Anosov diffeomorphisms). It was shown that supported on Λ is a unique f-invariant Borel probability measure μ characterized by any one of the following four equivalent conditions:

(i) the conditional measures of μ on unstable manifolds are smooth,
(ii) $\mathfrak{m}$-a.e. $x \in \mathcal{B}(\Lambda)$, the basin of attraction of Λ, is generic with respect to μ (see definitions below),
(iii) $h_\mu(f) = \int \log|\det(Df|_{E^u})| d\mu$,
(iv) μ is the zero-noise limit of large classes of small random perturbations of f.

The measure μ above is called the *SRB measure*. The importance of this class of invariant measures was first recognized by Sinai and Ruelle, who constructed these measures for Anosov systems and Axiom A attractors, respectively, in [Si2, R1]; see also [BR, B]. These papers contain the ideas in (i)–(iii), though we have formulated some of them a little differently. (iv) was first proved by Kifer [Ki1]; see also [Y2].

We elaborate on the meaning of these four conditions: (i) is a geometric characterization of the measure; since in most cases there can be no invariant measures with densities as explained above, (i) is as close to having a density as μ can come. The equivalence of (i) and (iii) for uniformly hyperbolic attractors is what motivated the last two results in theorem 1.

With regard to (ii), we say $x \in M$ is *generic* with respect to μ if for every continuous observable $\varphi : M \to \mathbb{R}$,

$$\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i x) \to \int \varphi \, \mathrm{d}\mu \qquad \text{as } n \to \infty, \tag{4}$$

i.e. starting from x, time averages converge to the space average. It is important to understand the distinction between (ii) and the Birkhoff ergodic theorem. The ergodic theorem states that if μ is ergodic, then time averages converge to the space average for μ-a.e. initial condition, while (ii) asserts this convergence for $\mathfrak{m}$-a.e. point in the open set $\mathcal{B}(\Lambda) := \{y \in M : d(f^n y, \Lambda) \to 0 \text{ as } n \to \infty\}$, even when μ is singular. The authors of [ER] termed an invariant measure with a positive $\mathfrak{m}$-measure set of generic points *physically relevant*. The motivation for (iv) is that the world is inherently noisy, and if μ_ε is the stationary measure for noise level ε, then $\lim_{\varepsilon\to 0}\mu_\varepsilon$ is in some sense the invariant measure that best describes what one observes.

We explain next how (i) implies (ii): pick a W^u_{loc}-leaf W with the property that $W' := \{x \in W : x \text{ is generic with respect to } \mu\}$ has full $\mathfrak{m}_W$-measure; here, $\mathfrak{m}_W$ is the induced Riemannian measure on W, and condition (i) states that almost all unstable manifolds have this property. Observe that if $x \in W'$, then all $y \in W^s_{\mathrm{loc}}(x)$ are generic since $d(f^i x, f^i y) \to 0$ and the test functions are continuous. Thus, all $y \in V' = \cup_{z\in W'} W^s_{\mathrm{loc}}(z)$ are generic. By the absolute continuity of the W^s-foliation (see, e.g., [An]), V' is a full Lebesgue measure subset of the open set $V = \cup_{z\in W} W^s_{\mathrm{loc}}(z)$. This proves (ii), as sets of the type V cover a neighborhood of Λ.

Positive LE also follows from similar reasoning. Here, in fact, $\lambda_{\max}(y) > 0$ for every $y \in \mathcal{B}(\Lambda)$. This is because $y \in W^s(x)$ for some $x \in \Lambda$, and $\lambda_{\max}(x) > 0$ because Λ is a uniformly hyperbolic attractor. The positivity of $\lambda_{\max}(y)$ follows from the fact that any two points x and y with the property that $d(f^i x, f^i y) \to 0$ exponentially fast have the same Lyapunov exponents.

Some of the ideas for uniformly hyperbolic attractors were generalized in the 1980s by Ledrappier, Young and others. First, SRB measures were constructed for piecewise uniformly hyperbolic attractors and shown to have some of the properties above [Y1]. Then, (i) and (iii) above were shown to be equivalent for all diffeomorphisms and all invariant measures [LS, L, LY1], legitimizing the idea that the concept of SRB measures as defined in section 2.2 makes sense for general dynamical systems. Another important step forward is the extension of the result on the absolute continuity of stable foliations to the nonuniform hyperbolic setting [PuSh]. This implies, by an argument similar to that for '(i)⇒(ii)' above, that whenever a system admits an ergodic SRB measure μ with no zero Lyapunov exponents, the set of points y that lie in $W^s(x)$ for some μ-typical x has positive $\mathfrak{m}$-measure. That is to say, μ is physically relevant. The same reasoning gives positive LE and hence observable chaos.

A missing ingredient in this expanded theory is that questions related to the *existence* of SRB measures were unsettled—and these questions have remained open to this day. Indeed not all attractors admit SRB measures; it is not enough to be hyperbolic on large parts of the phase space (see [HY] for an example that is not hyperbolic at only one point). In the next section, we will explain why in general it is very hard to analytically determine if a given system has positive LE.

5. Proving positive LE

5.1. Systems with and without invariant cones

Uniform hyperbolicity is generally established through the identification of *invariant cones*. More precisely, to show that an invariant set Λ is uniformly hyperbolic, it suffices to identify

(a) a continuous family of unstable cones $\{\mathcal{C}^u(x), x \in \Lambda\}$, one in the tangent space at each point in Λ, with the properties that (i) $Df_x(\mathcal{C}^u(x)) \subset \mathcal{C}^u(fx)$ and (ii) for some $\kappa > 1$, $|Df(v)| \geqslant \kappa|v|$ for $v \in \mathcal{C}^u$, and (b) a family of stable cones $\{\mathcal{C}^s\}$ satisfying (i) and (ii) above with f replaced by f^{-1}. Uniform hyperbolicity is equivalent to the existence of these cone families, and this 'invariant cones condition' is generally not hard to check because it is robust under perturbations.

There are several variants of this cones condition. We mention two that have been especially fruitful. One is the generalization from uniform hyperbolic to (uniform) *partially hyperbolic* systems defined by the existence of a continuous splitting of the tangent space into $E^u \oplus E^c \oplus E^s$ (i.e. three invariant cones) where the action of $Df|_{E^c}$ is between those on E^u and E^s, i.e. vectors in E^c can be expanded or contracted but not as strongly as those in E^u or E^s (see, e.g., [PeSi, ShPu]). In another generalization, condition (ii) in the last paragraph is dropped and replaced by a condition requiring that for a.e. x, $\mathcal{C}^u(x)$ be mapped under Df_x^n for some $n > 0$ strictly into the interior of $\mathcal{C}^u(f^n x)$. This idea was proposed in [Wo1] and applied successfully to prove positive LE in large classes of billiards and bouncing ball systems; see, e.g., [Wo2].

A good fraction of the analytical work in hyperbolic theory in the last 20–30 years has been based on the assumption of invariant cones, due in part to the relative tractability of such systems and in part to motivation from systems like billiards and hard balls. A case in point is [LiW], which extends Sinai's proof of ergodicity [Si1] to general piecewise smooth Hamiltonian systems with invariant cones. Away from these important and natural examples, however, this is a rather special condition: most dynamical systems in the world do not have invariant cones.

The rest of this section is about systems *without* invariant cones. We will discuss some of the challenges that one faces when attempting to establish the positivity of Lyapunov exponents in such systems.

Geometric expansion or stretching in phase space is a pre-condition for positive LE: when $|Df_x(v)| \leqslant |v|$ everywhere, there can be no positive exponents to speak of. The presence of such expansion, however, does not imply positive LE, for expansion is necessarily accompanied by contraction, either in different directions at the same points or elsewhere in the phase space—unless there is a net volume increase (which is not possible for volume-preserving systems or near attractors). This means that unless expanding and contracting directions are well separated and consistently aligned, i.e. unless there are invariant cones, the sequence $|Df_x^n(v)|$, $n = 1, 2, \ldots$, for a typical vector will sometimes go up and sometimes go down, and whether $\lambda(x, v)$ eventually ends up positive or negative is the result of a delicate balance. It is a trajectory-dependent cancellation problem that is very hard to control, and determining whether or not a system has positive LE involves understanding this problem for positive measure sets of initial conditions.

Here is another way to understand how this cancellation problem comes about. Suppose we observe that there are disjoint stretches of time $[n_i, n_i + s_i]$, $i = 1, 2, \ldots$, on which an orbit behaves hyperbolically, say $|Df^{s_i}_{f^{n_i}x} v_i| \geqslant e^{cs_i}|v_i|$ for some $c > 0$. This still does not imply $\lambda(x, v) > 0$ for any v, no matter how long these finite-time intervals are, for the vectors v_i, which are expanded in the ith interval, may get contracted in subsequent times. On a more formal level, one can attribute this to the *submultiplicativity* of matrix norms: given two matrices A and B, we have only $\|AB\| \leqslant \|A\|\|B\|$, not equality, and $\|AB\|$ can be significantly smaller than $\|A\|\|B\|$ for the reasons just explained.

We mentioned earlier that positive LE follows once we know of the existence of an SRB measure with no zero exponents. Not surprisingly, proving the existence of SRB measures involves similar difficulties: it requires not only that there be directions of sustained expansion, but that these directions had to be suitably aligned, so that at the end of the construction they are tangent to unstable manifolds. Still, to my knowledge, all known cases of positive LE without invariant cones have been obtained through the construction of SRB measures; the two tend to go hand-in-hand.

5.2. Three case studies

We now provide evidence to support our contention that the cancellation problems discussed in section 5.1 are real, meaning they really do occur. Three case studies are presented. These studies show complicated dynamical landscapes as parameters are varied, in contrast to the case of uniformly hyperbolic attractors, which are structurally stable (hence the picture does not change). We believe that complicated dynamical landscapes are the typical state of affairs. Indeed, what we see below is likely the tip of the iceberg, for the systems in all three studies are low dimensional and systems with many degrees of freedom are capable of exhibiting far greater complexity.

The logistic family. This 1-parameter family provides the simplest examples of genuinely nonuniformly hyperbolic maps. It is given by

$$f_a : [-1, 1] \circlearrowleft, \quad \text{where} \quad f_a(x) = 1 - ax^2, \ \ a \in [0, 2].$$

For a near 2, $|f_a'| > 1$ on a large part of $[-1, 1]$, but decreases to 0 at $x = 0$. For these parameters, the sequence of derivatives $|(f^n)'(x)|, n = 1, 2, \ldots$, for a typical x simulates to some degree the rises and falls of $|Df_x^n(v)|$ in section 5.1: away from 0, the derivative increases, and the closer to 0 the orbit comes, the larger the drop. Whether or not f_a has positive LE depends on the balance of these rises and falls, and as the following results show, the situation is far from simple. It has been proved that (a) for an open and dense set of parameters a, the orbit of Leb-a.e. x tends to a sink [GS, Ly1], while (b) for a positive Lebesgue measure set of a, f_a has an invariant density and positive LE [J]. See also [Ly2]. The maps in (a) and (b) have diametrically opposite properties, yet their parameter sets are highly entangled. Try to imagine how the positive measure set of parameters in (b) is nestled in the complement of the open and dense set of parameters in (a), and one will begin to get an appreciation of the delicateness of the situation.

Hénon maps and, more generally, rank-1 attractors. The next simplest examples are given by the Hénon family

$$T_{a,b} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \text{with} \qquad T_{a,b}(x, y) = (1 - ax^2 + y, \ bx).$$

For $b \ll 1$ and $a \approx 2$, Benedicks and Carleson [BC] were able to successfully control the cancellation problem for a positive measure set of parameters. Building on that, Benedicks and Young constructed SRB measures and proved positive LE for the same parameters [BY].

Not long thereafter, Wang and Young extended this body of ideas, including the existence of SRB measures and positive LE, to a much larger class of attractors called *rank-1 attractors*. These attractors can live in phase spaces of any dimension, but have exactly one unstable direction [WY1, WY3]. Being (in the sense of having only one unstable direction) the least chaotic of strange attractors, rank-1 attractors tend to emerge after a system loses its stability. A number of natural examples are of this type, among them are strange attractors arising from periodically kicked oscillators and systems undergoing supercritical Hopf bifurcations; see [WY2, LWY].

For rank-1 attractors (including the Hénon attractors), SRB measures and positive LE are proved only for positive measure sets of parameters. Periodic sinks, which are known to occur near tangencies of stable and unstable manifolds [N], are found in open regions of parameter space between parameters with positive LE.

A readily accessible example with a rich and varied dynamical landscape is the periodically kicked linear shear flow (reviewed in [LiY]). This equation has three parameters, representing the strengths of the shear, damping and kick size. Depending on parameter choice, the time-T map (T = kick period) can be in one of the following regimes: (i) quasi-periodic dynamics on a closed invariant curve, (ii) periodic sinks on invariant curve, (iii) gradient-like dynamics, (iv) horseshoes and sinks, or (v) rank-1 strange attractors with SRB measures. Regimes (i) and (ii) compete in certain parameter regions; the transition from (i) or (ii) to (iv) via (iii) is messy, and (iv) and (v) compete with wildly entangled parameter sets.

Standard maps. This family of area-preserving maps of the two-dimensional torus $\mathbb{T}^2$ is defined by

$$f_k(x, y) = (x + y + k \sin(2\pi x), y + k \sin(2\pi x)),$$

where $k > 0$ is a parameter. For k large, f_k has strong expansion and contraction, their directions separated by clearly defined invariant cones on most of the phase space—except for narrow strips (corresponding to the critical points of the sine function) on which the expanded vectors 'turn around' violating the cone preservation. The areas of these 'critical regions' tend to zero as $k \to \infty$. Despite the strong hyperbolic properties that are clearly visible, the question of whether or not any f_k, for k however large, has positive LE has remained unresolved, at least mathematically. The closest to positive LE that has been proved is that the set of points with positive Lyapunov exponents has Hausdorff dimension 2 [Go], a condition much weaker than positive LE. It is also known that for a residual set[1] of parameters k, f_k has many elliptic islands [D, Go]. Although elliptic islands and positive LE are not mutually exclusive, the presence of islands is testimony to the fact that the cancellations discussed in section 5.1 do occur.

Conclusion from case studies. These studies confirm that for systems without invariant cones, parameter space is partitioned into sets representing competing dynamical regimes, and these parameter sets can be wildly entangled. Given a parameter value, then, it is hard to determine exactly to which set it belongs. Suppose one wishes to determine if a given concrete system has a sink or positive LE. If a sink is observed within a finite number of iterates, then there are ways to confirm its presence (using, e.g., the contraction mapping theorem). But if that conclusion is not reached after a finite number of iterates, no matter how large, one still cannot conclude with certainty that the system has positive LE, because to do so would require knowledge of *an infinite number of iterates* at *infinite precision*; equivalently, it takes an infinite amount of information to conclude definitively that a given parameter value lies in a positive measure set (with empty interior). This, of course, is a purely theoretical issue; in practice, one does the best that one can.

6. Random dynamical systems

Most realistic systems are governed by laws that are neither purely deterministic nor purely stochastic but a combination of the two. In this section, we revisit some of the properties treated earlier and discuss them in the setting of noisy dynamical systems.

[1] A residual set is the intersection of a countable number of open and dense sets.

6.1. Setting and definitions

We consider in this section compositions of i.i.d. sequences of random maps. One motivation for this setup is that noise terms are routinely added to differential equations to model uncontrolled fluctuations or forces not accounted for, and it is known that solutions of stochastic differential equations, such as

$$\mathrm{d}x_t = a(x_t)\,\mathrm{d}t + \sum_{i=i}^{n} b_i(t) \circ \mathrm{d}W_t^i,$$

where W_t^i is white noise, have representations as *stochastic flows of diffeomorphisms*, i.e. for each ω corresponding to a realization of Brownian path, there is a 1-parameter family of diffeomorphisms $x \mapsto \varphi_t(x;\omega)$ satisfying $\varphi_{s+t}(x;\omega) = \varphi_t(\varphi_s(x;\omega);\sigma_s(\omega))$, where σ_s is time-shift along the path. See, e.g., [Ku]. Thus, systems modeled by SDEs can be seen as *i.i.d.* sequences of random maps, which appear also in other situations including algorithms involving random choices. These objects have been studied a fair amount; see, e.g., [Ki2, Ar].

The setting for the rest of this section is as follows. We consider compositions of the type

$$\cdots \circ f_n \circ \cdots \circ f_2 \circ f_1\,, \quad n = 1, 2, \ldots,$$

where $f_1, f_2, \ldots$ are chosen independently with respect to a probability ν on the space of diffeomorphisms of a compact manifold M. This defines a Markov chain on M with transition probabilities $P(x, A) = \nu\{f, f(x) \in A\}$. Let μ be a stationary measure on M, i.e. $\mu(A) = \int P(x, A)\mathrm{d}\mu(x)$. We will refer to this process as $\mathcal{X} = \mathcal{X}(M, \nu; \mu)$. Given $\mathcal{X}$, it is known that Lyapunov exponents are defined μ-a.e. for $\nu^{\mathbb{Z}^+}$-almost every sequence of maps, and these numbers are nonrandom. Let us denote them as before by $\lambda_1 > \lambda_2 > \ldots > \lambda_r$ with multiplicities m_i. Likewise, the pathwise entropy h is defined for almost every sequence and is nonrandom.

To state our results, we need to introduce the idea of *sample measures*. Viewing the process $\mathcal{X}$ as having started from time $-\infty$, one obtains a family of sample measures $\{\mu_{\underline{\mathbf{f}}}\}$ defined for $\nu^{\mathbb{Z}}$-a.e. $\underline{\mathbf{f}} = (f_i)_{i=-\infty}^{\infty}$ by conditioning μ on the past. That is to say, $\mu_{\underline{\mathbf{f}}}$ describes what one sees at time 0 given that the history of the system, i.e. $(f_i)_{i<0}$, is known. Put differently, μ and $\mu_{\underline{\mathbf{f}}}$ are related by

$$\mu = \int \mu_{\underline{\mathbf{f}}}\,\mathrm{d}\nu^{\mathbb{Z}}(\underline{\mathbf{f}}) \qquad \text{and} \qquad \mu_{\underline{\mathbf{f}}} = \lim_{n\to\infty} (f_{-1} \circ f_{-2} \circ \cdots \circ f_{-n})_*\mu.$$

Sample measures are pathwise invariant, in the sense that $(f_0)_*(\mu_{\underline{\mathbf{f}}}) = \mu_{\sigma\underline{\mathbf{f}}}$, where σ is the shift map, i.e. if $\underline{\mathbf{f}} = (f_i)$ and $\sigma\underline{\mathbf{f}} = (g_i)$, then $g_i = f_{i+1}$. See, e.g., [LY3] for a more systematic discussion.

6.2. Entropy formula and random attractors

It is reasonable to expect that with the averaging effects of random noise, events that occur as a result of 'coincidences' in purely deterministic systems will disappear, and the dynamical picture is nicer. Note for a start that genuinely random noise will produce a density for μ independently of the underlying dynamics: if $P(x,\cdot)$ has densities for all x, then μ has a density, and this is not a necessary condition.

Theorem 4. *The following hold for $\mathcal{X} = \mathcal{X}(M, \nu; \mu)$ assuming that the process is ergodic (in addition to the usual integrability conditions for derivatives).*

(1) [LeJ] If $\lambda_1 < 0$, then $\mu_{\underline{\mathbf{f}}}$ is supported on a finite set of points for $\nu^{\mathbb{Z}}$-a.e. $\underline{\mathbf{f}}$.

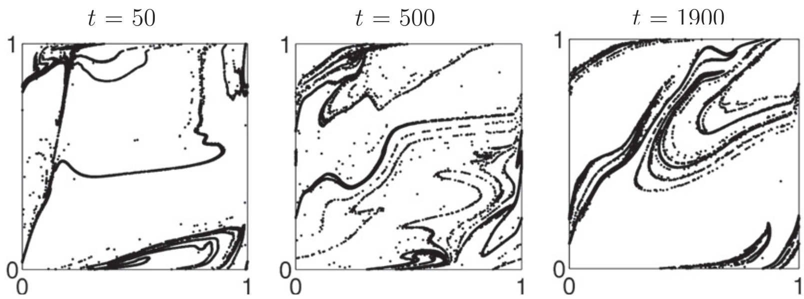

Figure 3. Snapshots of sample measures for two coupled phase oscillators driven by a white-noise stimulus. The phase space is the 2-torus, and parameters are chosen so that the system is unreliable. The curves seen are the unstable manifolds of random strange attractors (on which SRB measures are supported). The attractors evolve perpetually with time, retaining certain basic characteristics throughout.

(2) [LY2] If μ has a density, and $\lambda_1 > 0$, then

$$h = \sum_i \lambda_i^+ m_i,$$

and for $\nu^{\mathbb{Z}}$-a.e. $\underline{\mathbf{f}}$, $\mu_{\underline{\mathbf{f}}}$ is a random SRB measure, i.e. it has smooth conditional densities on unstable manifolds.

The two results in theorem 4 state that except where $\lambda_1 = 0$ (in which case the theorem offers no information), there is a simple dichotomy in the dynamical picture: either almost all solutions coalesce into at most a finite number of trajectories, which then evolve together in what is called a *random sink*, or the system has a *random strange attractor*, i.e. an attracting set which has all the attributes of the attractors with SRB measures discussed earlier except that they evolve with time. The simple characterizations of $\mu_{\underline{\mathbf{f}}}$, together with the fact that Lyapunov exponents vary continuously under mild conditions, contrast with the situation for single maps, for which the picture in parameter space can be very complicated as we have discussed.

One application of these ideas is to the *reliability* of dynamical systems. A stimulus is presented, and the system's initial response will depend on its internal state at the stimulus onset. The question is whether or not this dependence on initial state persists. If it does, then the system is inherently *unreliable*, in that its response to a given stimulus may vary from trial to trial. In situations where the stimulus has the form of a noise (modeling fluctuating input), this question can be viewed in the framework of stochastic flows, and reliability is equivalent to trajectories with different initial conditions coalescing into a single trajectory for a frozen Brownian noise, i.e. it has to do with the sign of λ_1. Reliability questions have repercussions in many biological and engineered systems. See, e.g., [LSbY] for a case study. Sample measures for an unreliable system are shown in figure 3.

6.3. Dimension formulae

We begin by recalling the Kaplan–Yorke conjecture, put forth in [FKYY]. In the setting of a diffeomorphism of a manifold M having an attractor with an SRB measure μ, the authors of [FKYY] conjectured that, pathological cases excepted, the dimension of the attractor is given by a quantity that they called *Lyapunov dimension*. In the notation of this paper, this quantity is defined as follows. Let K be the largest integer such that $\sum_{i=1}^{K} \lambda_i m_i > 0$. Then,

$$\mathrm{LyapDim} = \begin{cases} \dim(M), & \text{if } \displaystyle\sum_{i=1}^{K} m_i = \dim(M), \\ \displaystyle\sum_{i=1}^{K} m_i - \frac{1}{\lambda_{K+1}} \sum_{i=1}^{K} \lambda_i m_i, & \text{otherwise.} \end{cases}$$

We will discuss this conjecture in the context of random dynamical systems. Let $\mathcal{X} = \mathcal{X}(M, \nu; \mu)$ be the same as in section 6.1. Assuming $\mathcal{X}$ is ergodic, we let LyapDim($\mathcal{X}$) denote the quantity above where the λ_i's are those of $\mathcal{X}$.

LyapDim($\mathcal{X}$) will be related to other notions of dimension. In section 3.2, we defined $\dim(\mu)$, the *dimension* of a measure μ, and for a system (f, μ), we introduced the idea of *partial dimensions* δ_i in the directions of the invariant subspaces E_i corresponding to λ_i. We used only δ_i corresponding to $\lambda_i > 0$ in theorem 2, but δ_i can be defined also for $\lambda_i < 0$ by considering f^{-1}. Moreover, these ideas can all be extended to $\mathcal{X} = \mathcal{X}(M, \nu; \mu)$. Specifically, $\dim(\mu_{\underline{\mathbf{f}}})$ is well defined for a.e. $\underline{\mathbf{f}}$ and is nonrandom, as are $\delta_i \in [0, m_i]$ corresponding to $\lambda_i \neq 0$, referring to partial dimensions of $\mu_{\underline{\mathbf{f}}}$.

For the next result to hold, we consider $\mathcal{X} = \mathcal{X}(M, \nu; \mu)$, and in addition to requiring that μ have a density, we need to assume a technical condition that corresponds to diffusion for the backward derivative process associated with $\mathcal{X}$. Roughly speaking, this means that not only do the images of a point have to be random, the directions of tangent vectors have to be random as well[2]. We refer the reader to [LY4], as precise formulations are technical; suffice it to say here that this condition is satisfied by large classes of SDEs.

Theorem 5. *[LY4] Suppose $\mathcal{X} = \mathcal{X}(M, \nu; \mu)$ satisfies the conditions above and assume additionally that $\lambda_i \neq 0$ for all i. Then,*

$$\dim(\mu_{\underline{\mathbf{f}}}) = \sum_i \delta_i = \mathrm{LyapDim}(\mathcal{X}). \tag{5}$$

The second equality above is equivalent to the following: if we write $\delta_i = \sigma_i m_i$ so that $\sigma_i \in [0, 1]$, then there is a critical index i_c with the property that

$$\sigma_i = 1 \ \text{ for } \ i < i_c \qquad \text{and} \qquad \sigma_i = 0 \ \text{ for } \ i > i_c. \tag{6}$$

The first equality in (5) was first proved for SRB measures with no zero Lyapunov exponents in the purely deterministic setting [LY2], and it is valid in the random case for the same reason; see theorem 3. For individual maps, i.e. for arbitrary (f, μ), δ_i can assume various configurations subject to constraints; for example, one might consider measures that are products. The configuration of σ_i in (6), on the other hand, is very special. It suggests that when randomly perturbed or 'shaken', mass has a tendency to align with the more expanding directions, or at least it fills up the more expanding directions before getting to the less expanding ones.

This concludes our review. It is fair to say, in light of theorems 4 and 5, that the dynamical pictures for randomly perturbed systems are simpler and nicer than those for purely deterministic ones, as we have suggested at the beginning of section 6.2.

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[2] Here is one way to say it: let Gr(M) denote the Grassmannian bundle of M. Then, for any $v \in \mathrm{Gr}(M)$ and $\Gamma \subset \mathrm{Gr}(M)$, the transition probabilities $Q(v, \Gamma) = \nu\{f, (Df^{-1}v \in \Gamma\}$ have a density.

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