1. $\vec{F}(x,y,z) = (x,y,z)$. Prove there doesn't exist a vector field $\vec{G}$ such that $\vec{F} = \vec{\nabla} \times \vec{G}$.

   Solution: Observe that $\vec{\nabla}.\vec{F} = 3$, and we know $\vec{\nabla}.(\vec{\nabla} \times G) = 0$ for any $\vec{G}$, so $\vec{F}$ is not the curl of any vector field.

2. Compute the volume of the solid $E = \{(x,y,z) \in \mathbb{R}^3 : x^2+z^2 \leq 1, y^2+z^2 \leq 1\}$

   Solution: The xy-plane projection of E is the square $[-1,1] \times [-1,1]$. Let $1-x^2 \leq 1-y^2$, we get $|x^2| \geq |y|^2$. So E is bounded above and below by $z = \pm\sqrt{1-x^2}$ on the region $\{(x,y) \in \mathbb{R}^2 | 0 \leq x \leq 1, -x \leq y \leq x\}$. So the volume can be computed by symmetry:

$$V(E) = 4\int_0^1 \int_{-x}^{x} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz\,dy\,dx = \frac{16}{3}$$

3. $f(x,y,z)$ is a scalar function and $\vec{F}(x,y,z)$ is a vector field. Prove

$$\vec{\nabla}.(f\vec{F}) = f(\vec{\nabla}.\vec{F}) + (\vec{\nabla}f).\vec{F}$$

   Solution: Let $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$.

$$\begin{aligned}
&\vec{\nabla}.(f\vec{F})\\
=&\vec{\nabla}.(fP, fQ, fR)\\
=&(fP)_x + (fQ)_y + (fQ)_z\\
=&fP_x + f_xP + fQ_y + f_yQ + fR_z + f_zR\\
=&f(P_x + Q_y + R_z) + (f_xP + F_yQ + F_zR)\\
=&f(\vec{\nabla}.\vec{F}) + (\vec{\nabla}f).\vec{F}
\end{aligned}$$

4. Let E be the unit cube $E = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\}$. Let S be the boundary of E with outward orientation. If $\vec{F}(x,y,z) = (2xy, 3ye^z, x\sin z)$, compute $\oiint_S \vec{F}.\,d\vec{S}$

Solution: By divergence theorem,

$$\begin{aligned}\oiint_S \vec{F}.\,d\vec{S} &= \iiint_E \vec{\nabla}.\vec{F}\,dV \\ &= \int_0^1\int_0^1\int_0^1 2y + 3e^z + x\cos z\,dx\,dy\,dz \\ &= 3e + \frac{1}{2}\sin z - 2\end{aligned}$$

5. $\vec{F}(x,y,z)$ is a vector field defined on $\mathbb{R}^3$. $0 < r < R$ are constants. Let S_R (S_r) denote the sphere of radius R (r) centered at origin. E is the region bounded by these two spheres. Prove

$$\iiint_E div\vec{F}\,dV = \oiint_{S_R} \vec{F}.\,d\vec{S} - \oiint_{S_r} \vec{F}.\,d\vec{S}$$

Solution: Let $B(R)(B(r))$ be the ball enclosed by the sphere $S_R(S_r)$. By the Divergence Theorem,

$$\begin{aligned}&\oiint_{S_R} \vec{F}.\,d\vec{S} - \oiint_{S_r} \vec{F}.\,d\vec{S} \\ &= \iiint_{B(R)} div\vec{F}\,dV - \iiint_{B(r)} div\vec{F}\,dV \\ &= \iiint_E div\vec{F}\,dV\end{aligned}$$