Example. $F(x,y,z) = x^2+y^2+z^2$. $F(x,y,z) = C > 0$ is a level set on which $\nabla F(x,y,z) = (2x, 2y, 2z) \neq 0$. so

$F(x,y,z) = C$ is a differentiable surface.

Indeed we know it represents a sphere

Exercise. $F(x,y) = xy$. $F(x,y) = C$ is a differentiable curve if $C \neq 0$. And what happened for $C=0$?

We can use the formula in the Implicit Function Theorem to compute the tangent line / tangent surface

Example. Find the tangent line of $\frac{x^2}{4} + \frac{y^2}{9} = 1$ at $(1, \frac{3}{2}\sqrt{3})$

$F(x,y) = \frac{x^2}{4} + \frac{y^2}{9}$ $\nabla F(x,y) = (\frac{x}{2}, \frac{2}{9}y)$

we see $\frac{2}{9} \cdot \frac{3}{2}\sqrt{3} \neq 0$. so locally $y = g(x)$ on $F(x,y) = 1$, i.e. $F(x, g(x)) = 1$.

then $$g'(x) = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{\frac{x}{2}}{\frac{2}{9}y} = -\frac{9x}{4y}$$

$g'(1) = -\frac{\sqrt{3}}{2}$, the tangent line is

$$y - \frac{3}{2}\sqrt{3} = -\frac{\sqrt{3}}{2}(x-1)$$

Theorem $F:\mathbb{R}^3 \longrightarrow \mathbb{R}$ is a continuously differentiable function. If $F^{-1}(c)$ defines a surface, i.e. $\nabla F(x,y,z)\neq\vec{0}$ for any $F(x,y,z)=c$, then $\nabla F(x,y,z)$ is a normal vector field on the surface.

Proof. Without loss of generality, assume $\frac{\partial F}{\partial z}(x_0,y_0,z_0)\neq 0$. Then by Implicit Function Theorem, there's $\mathbb{R}^2 \supseteq A \xrightarrow{G} B \subseteq \mathbb{R}$, $(x_0,y_0)\in A$, $z_0\in B$ such that $F(x,y,G(x,y))=c$ on A.

So locally the surface is the graph of $z=G(x,y)$.

$$\nabla G(x_0,y_0) = -\frac{1}{\frac{\partial F}{\partial z}(x_0,y_0,z_0)}\cdot\left[\frac{\partial F}{\partial x}(x_0,y_0,z_0), \frac{\partial F}{\partial y}(x_0,y_0,z_0)\right]$$

We know $\left(-\frac{\partial G}{\partial x}(x_0,y_0,z_0), -\frac{\partial G}{\partial y}(x_0,y_0,z_0), 1\right)$ is a normal vector to the surface at (x_0,y_0,z_0), and

$$\left(-\frac{\partial G}{\partial x}(x_0,y_0,z_0), -\frac{\partial G}{\partial y}(x_0,y_0,z_0), 1\right) = \frac{1}{\frac{\partial F}{\partial z}(x_0,y_0,z_0)}\nabla F(x_0,y_0,z_0)$$

So $\nabla F(x_0,y_0,z_0)$ is also a normal vector.

Example Find the equation of the tangent plane of $x^2+2y^2+3z^2=6$ at $(1,1,1)$.

Let $F(x,y,z)=x^2+2y^2+3z^2$.

$\nabla F(x,y,z)=(2x,4y,6z)$, so $\nabla F(1,1,1)=(2,4,6)$

The equation of the tangent plane is

$$2(x-1)+4(y-1)+6(z-1)=0$$

Similar to the discussions about curves in $\mathbb{R}^2$ and surfaces in $\mathbb{R}^3$, we can also study curves in $\mathbb{R}^3$ by the implicit function theorem.

Proposition. $F: \mathbb{R}^3 \to \mathbb{R}^2$ is a smooth map. $F(x,y,z)=(C_1,C_2)$ a level set on which DF is of rank 2 everywhere. then $F(x,y,z)=(C_1,C_2)$ represents a differentiable curve in $\mathbb{R}^3$

Example: $F: \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $F(x,y,z)=(xy, yz+xz)$

$$DF = \begin{pmatrix} y & x & 0 \\ z & z & x+y \end{pmatrix}$$

We see $DF(1,1,1) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$ with

$\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ invertible, so near $(1,1,1)$ there's

function $G: \mathbb{R} \to \mathbb{R}^2$ such that $F(x, G(x)) = (1, 2)$

So near $(1,1,1)$ $F(x,y,z)=(1,2)$ is the curve $\vec{r}(t)=(t, G(t))$ for t near 1., and we can compute $\vec{r}\,'(1)$ by $\vec{r}\,'(t) = (1, G'(t))$

$$G'(1) = -\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

So $\vec{r}\,'(1) = (1, -1, 0)$

APPLICATIONS IN ECONOMICS

Example. The demand D for a goods is a functions of the price P before tax and the sales tax is t per unit:

$$D = f(t, p)$$

Suppose S is the supply, which is a function of price P:

$$S = g(p)$$

The equilibrium price p is a function of the sales tax t, implicitly defined by

$$f(t, p) = g(p)$$

Let $F(t, p) = f(t, p) - g(p) = 0$

$DF = \left[\frac{\partial f}{\partial t} \quad \frac{\partial f}{\partial p} - \frac{dg}{dp}\right]$, so by the implicit function theorem the equilibrium price p is a function of t if $\frac{\partial f}{\partial p} - \frac{dg}{dp} \neq 0$

In reality, $\frac{\partial f}{\partial p} < 0$ and $\frac{dg}{dp} > 0$, so $\frac{\partial f}{\partial p} - \frac{\partial g}{\partial p} \neq 0$.

$$\frac{dP}{dt} = -\frac{\frac{\partial f}{\partial t}}{\frac{\partial f}{\partial p} - \frac{dg}{dp}} < 0,$$

which implies a increase in sales tax will make the equilibrium of price before tax decrease.

Example. A firm produces $Q = f(L)$ units of goods when using L units of labour. Assume $f'(L) > 0$, $f''(L) < 0$.

If the firm gets P dollars per unit produced and pays w dollars for a unit of labour. Then the profit function is given by:

$$\pi(L) = \underbrace{Pf(L)}_{\text{revenue}} - \underbrace{wL}_{\text{cost}}$$

Let L^* be the units of Labour that maximizes $\pi(L)$. We know

$$\pi'(L^*) = 0$$

i.e. $$pf'(L^*) - w = 0$$

Let $F(p, w, L^*) = pf'(L^*) - w = 0$

$$\nabla F = [f'(L^*), -1, pf''(L^*)]$$

Since $pf''(L^*) \neq 0$, L^* is a function of p & w.
$L^* = G(p, w)$, $pf'(G(p,w)) - w = 0$

$$\left[\frac{\partial L^*}{\partial p}, \frac{\partial L^*}{\partial w}\right] = -\frac{1}{pf''(L^*)}[f'(L^*), -1]$$

$$= \left[-\frac{f'(L^*)}{pf''(L^*)}, \frac{1}{pf''(L^*)}\right]$$

We see as price increases, L^* increases.
as salary increases, L^* decreases