DUAL SPACES AND BILINEAR FORMS

Definition. V is a vector space over a field $\mathbb{F}$. A map $T: V \longrightarrow \mathbb{F}$ is defined to be a linear map if $\forall\, \vec{u}, \vec{v} \in V$, $\lambda, \mu \in \mathbb{F} \Rightarrow T(\lambda\vec{u} + \mu\vec{v}) = \lambda T(\vec{u}) + \mu T(\vec{v})$

Given a vector space V over a field $\mathbb{F}$. Denote V^* to be the set of all linear maps $V \longrightarrow \mathbb{F}$. We can define an addition and a scalar multiplication on V^*:

(i) (Addition) $\forall\, T_1, T_2 \in V^*$, $T_1 + T_2$ is the linear map

$$(T_1 + T_2)(\vec{v}) = T_1(\vec{v}) + T_2(\vec{v}) \quad \forall\, v \in V.$$

(ii) (Scalar Multiplication) $\forall\, T \in V^*$, $\lambda \in \mathbb{F}$, λT is the linear map $(\lambda T)(\vec{v}) = \lambda T(\vec{v}) \quad \forall\, v \in V$.

Definition. The dual vector space of V is the vector V^* with addition and scalar multiplication defined as above.

In case when the vector space is of finite dimension, there's a good way to represent linear maps:

One way to understand a linear map is to regard it as a linear transformation from $\mathbb{F}^n$ to $\mathbb{F}^1$. So it will be represented by a $1 \times n$ matrix (i.e. a row vector $(a_1, \dots, a_n)$. If $\varphi = (a_1, \dots, a_n)$, then $\varphi(\vec{v}) = (a_1, \dots, a_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = a_1 v_1 + \cdots + a_n v_n$,

and $a_i = \varphi(\vec{e}_i)$. This observation motivates the following:

Lemma. If V is a finite dimensional vector space over $\mathbb{F}$, then V^* is also a finite dimensional vector space over $\mathbb{F}$, and $\dim V^* = \dim V$

Proof. If we take a basis $\{v_1, v_2, \ldots, v_n\}$ for V, where $n = \dim V$, define $\varphi_i \in V^*$ to be the linear map satisfying

$$\varphi_i(\vec{v}_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Then we'll show $\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ forms a basis of V^* and we call it the dual basis of $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$

First, we show $\varphi_1, \ldots, \varphi_n$ are linearly independent:
If $\lambda_1\varphi_1 + \cdots + \lambda_n\varphi_n = 0$ for some $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$.
For any $\vec{e}_i$, $(\lambda_1\varphi_1 + \cdots + \lambda_n\varphi_n)(\vec{v}_i) = 0$

$$\Rightarrow \quad \lambda_i \varphi_i(\vec{v}_i) = 0$$

$$\Rightarrow \quad \lambda_i = 0 \quad \text{since } \varphi_i(\vec{v}_i) = 1$$

So $\lambda_1 = \cdots = \lambda_n = 0$. we see $\varphi_1, \ldots, \varphi_n$ are linearly independent.

Next, we show $\operatorname{span}\{\varphi_1, \ldots, \varphi_n\} = V^*$:
For any $\varphi \in V^*$, let $\mu_i = \varphi(\vec{v}_i)$, then we see

$$\varphi(\vec{v}_i) = \mu_i = (\mu_1\varphi_1 + \cdots + \mu_n\varphi_n)(\vec{v}_i)$$

i.e. φ and $\mu_1\varphi_1 + \cdots + \mu_n\varphi_n$ agree on the basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$, so by linearity, $\varphi = \mu_1\varphi_1 + \cdots + \mu_n\varphi_n$

We conclude $\{\varphi_1, \ldots, \varphi_n\}$ is a basis for V^*

Definition. V is a vector space over $\mathbb{F}$. A bilinear form on V is a map
$$V \times V \xrightarrow{\langle\, , \,\rangle} \mathbb{F}$$
$$(\vec{u}, \vec{v}) \longmapsto \langle \vec{u}, \vec{v} \rangle$$
such that (i) $\forall\, \lambda, \mu \in \mathbb{F}$, $\vec{u}, \vec{v}, \vec{w} \in V$
$$\langle \lambda\vec{u} + \mu\vec{v}, \vec{w} \rangle = \lambda\langle \vec{u}, \vec{w} \rangle + \mu\langle \vec{v}, \vec{w} \rangle$$
(ii) $\forall\, \lambda, \mu \in \mathbb{F}$, $\vec{u}, \vec{v}, \vec{w} \in V$.
$$\langle \vec{w}, \lambda\vec{u} + \mu\vec{v} \rangle = \lambda\langle \vec{w}, \vec{u} \rangle + \mu\langle \vec{w}, \vec{v} \rangle$$

Example The dot product on $\mathbb{R}^n$ is a bilinear form:
$$(\lambda\vec{u} + \mu\vec{v}) \cdot \vec{w} = \lambda\vec{u}\cdot\vec{w} + \mu\vec{v}\cdot\vec{w}$$
$$\vec{w}\cdot(\lambda\vec{u} + \mu\vec{v}) = \lambda\vec{w}\cdot\vec{u} + \mu\vec{w}\cdot\vec{v}$$

Definition
- A bilinear form is symmetric if $\forall\, \vec{u}, \vec{v} \in V$, $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
- A bilinear form is nondegenerate if $\langle \vec{u}, \vec{v} \rangle = 0 \ \forall\, \vec{v} \in V$ implies $\vec{u} = \vec{0}$.

There is a linear transformation $V \longrightarrow V^*$ corresponding to a given bilinear form $\langle\, , \,\rangle$ on V:
$$V \longrightarrow V^*$$
$$\vec{u} \longmapsto \langle \vec{u}, \cdot \rangle$$

Proposition The map $V \longrightarrow V^*$ induced by a bilinear form $\langle\, , \,\rangle$ is injective if and only if $\langle\, , \,\rangle$ is nondegenerate.

Proof. Follows directly from the definition

A good way to represent a bilinear form is to use matrices.

Given a bilinear form $< , >$ on a finite dimensional vector space V over $\mathbb{F}$, let $n = \dim V$.
Take a basis of V: $\{\vec{v}_1, \dots, \vec{v}_n\}$
let $a_{ij} = <\vec{v}_i, \vec{v}_j>$, we form an $n \times n$ matrix $A = (a_{ij})$
For any $\vec{x} = x_1 \vec{v}_1 + \dots + x_n \vec{v}_n$ and $\vec{y} = y_1 \vec{v}_1 + \dots + y_n \vec{v}_n$.
We can see

$$<\vec{x}, \vec{y}> = (x_1, \dots, x_n) A \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

by verifying for $<\vec{v}_i, \vec{v}_j>$ then extend by linearity:

$$<\vec{v}_i, \vec{v}_j> = a_{ij} = (0, \dots, 0, \underset{\substack{\uparrow \\ i\text{-th}}}{1}, 0, \dots, 0) \, A \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \\ \\ \leftarrow j\text{-th} \\ \\ \\ \\ \end{matrix}$$

So once we fix a basis of V, there's a one-to-one correspondence between bilinear forms on V and $n \times n$ matrices with entries in $\mathbb{F}$

So now we've seen a new point of view for vectors and matrices: When we fix a basis for an n-dimensional vector space V over a field $\mathbb{F}$, a vector $(a_1, \dots, a_n)$ describes a linear map $V \to \mathbb{F}$ and a matrix (a_{ij}) describes a bilinear form (map) $V \times V \to \mathbb{F}$. This point of view indicates we can generalize the concepts of vectors and matrices by considering multilinear maps:

$$\underbrace{V \times V \times \dots \times V}_{m \text{ copies}} \longrightarrow \mathbb{F}.$$