Proposition. $\varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

Proof. You may refer to the textbook for the proof.

Example. Prove $\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$
\begin{aligned}
(\vec{a}\times(\vec{b}\times\vec{c}))_i &= \varepsilon_{ijk}\, a_j\, (\vec{b}\times\vec{c})_k \\
&= \varepsilon_{ijk}\, a_j\, \varepsilon_{klm}\, b_l\, c_m \\
&= \varepsilon_{kij}\, \varepsilon_{klm}\, a_j\, b_l\, c_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, a_j\, b_l\, c_m \\
&= (\delta_{jm}a_j)(\delta_{il}b_l)c_m - (\delta_{jl}a_j)\, b_l\, (\delta_{im}c_m) \\
&= a_m c_m b_i - a_l b_l c_i \\
&= ((\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c})_i
\end{aligned}
$$

SUFFIX NOTATION IN VECTOR CALCULUS

We can make use of the suffix notation to represent the differential operator $\vec{\nabla} = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ by

$$\vec{\nabla}_i = \frac{\partial}{\partial x_i}$$

So the gradient of f, $\vec{\nabla} f$ in suffix notation is

$$(\vec{\nabla} f)_i = \frac{\partial f}{\partial x_i}$$

the curl of $\vec{u}$, $\vec{\nabla} \times \vec{u}$ in suffix notation is

$$(\vec{\nabla} \times \vec{u})_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} u_k = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$$

the divergence of $\vec{u}$, $\vec{\nabla} \cdot \vec{u}$ in suffix notation is

$$\vec{\nabla} \cdot \vec{u} = \frac{\partial}{\partial x_i} u_i = \frac{\partial u_i}{\partial x_i}$$

Example. $\vec{r}(x_1, x_2, x_3) = (x_1, x_2, x_3)$. Let $r = |\vec{r}| = (\vec{r} \cdot \vec{r})^{\frac{1}{2}}$.

First, we know that $\frac{\partial x_i}{\partial x_j} = \delta_{ij}$ by the rules of differentiation.

$$[\vec{\nabla} r]_i = \frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i} (\vec{r} \cdot \vec{r})^{\frac{1}{2}} = \frac{\partial}{\partial x_i} (r_j \cdot r_j)^{\frac{1}{2}} = \frac{\partial}{\partial x_i} (x_j \cdot x_j)^{\frac{1}{2}}$$

$$= \frac{1}{2} (x_j \cdot x_j)^{-\frac{1}{2}} \cdot \frac{\partial}{\partial x_i} (x_j \cdot x_j)$$

$$= \frac{1}{2r} \cdot 2 x_j \frac{\partial x_j}{\partial x_i}$$

$$= \frac{1}{r} x_j \delta_{ij}$$

$$= \frac{1}{r} x_i$$

This implies

$$\vec{\nabla} r = \frac{\vec{r}}{r}$$

Next let's try to compute the curl and divergence of $\vec{r}$.

$$(\vec{\nabla}\times\vec{r})_i = \varepsilon_{ijk}\frac{\partial r_k}{\partial x_j} = \varepsilon_{ijk}\frac{\partial x_k}{\partial x_j} = \varepsilon_{ijk}\delta_{jk} = 0$$

so $\vec{\nabla}\times\vec{r} = \vec{0}$

$$\vec{\nabla}\cdot\vec{r} = \frac{\partial r_i}{\partial x_i} = \frac{\partial x_i}{\partial x_i} = \delta_{ii} = 3$$

The suffix notation can be used to verify the identities of vector calculus.

Example. Show $\text{Curl}(\nabla f) = \vec{0}$ and $\text{div}(\text{Curl}\,\vec{u}) = 0$

$$\begin{aligned}[\vec{\nabla}\times(\vec{\nabla}f)]_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{\nabla}f)_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\cdot\frac{\partial f}{\partial x_k} \\ &= -\varepsilon_{ikj}\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} \\ &= -\varepsilon_{ikj}\frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} \\ &= -[\vec{\nabla}\times(\vec{\nabla}f)]_i\end{aligned}$$

We conclude $\vec{\nabla}\times(\vec{\nabla}f) = \vec{0}$

$$\begin{aligned}\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) &= \frac{\partial}{\partial x_i}(\vec{\nabla}\times\vec{u})_i = \frac{\partial}{\partial x_i}\varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} \\ &= \varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial x_k}{\partial x_j} \\ &= -\varepsilon_{jik}\frac{\partial}{\partial x_j}\cdot\frac{\partial x_k}{\partial x_i} \\ &= -\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u})\end{aligned}$$

We conclude $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) = 0$

Example. Show that $\vec{\nabla}\times(\vec{\nabla}\times\vec{u}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \vec{\nabla}^2\vec{u}$

$$[\vec{\nabla}\times(\vec{\nabla}\times\vec{u})]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\vec{\nabla}\times\vec{u})_k$$

$$= \varepsilon_{ijk}\frac{\partial}{\partial x_j}\,\varepsilon_{klm}\frac{\partial u_m}{\partial x_l}$$

$$= \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial^2 u_m}{\partial x_j\,\partial x_l}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial^2 u_m}{\partial x_j\,\partial x_l}$$

$$= \delta_{il}\cdot\delta_{jm}\frac{\partial^2 u_m}{\partial x_j\,\partial x_l} - \delta_{im}\delta_{jl}\frac{\partial^2 u_m}{\partial x_j\,\partial x_l}$$

$$= \frac{\partial^2 u_j}{\partial x_j\,\partial x_i} - \frac{\partial^2 u_i}{\partial x_j\,\partial x_j}$$

$$= \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{\partial^2 u_i}{\partial x_j^2}$$

$$= [\vec{\nabla}(\vec{\nabla}\cdot\vec{u})]_i - [\vec{\nabla}^2\vec{u}]_i$$

Example. Show that $\vec{\nabla}\cdot(\vec{u}\times\vec{v}) = (\vec{\nabla}\times\vec{u})\cdot\vec{v} - (\vec{\nabla}\times\vec{v})\cdot\vec{u}$

$$\vec{\nabla}\cdot(\vec{u}\times\vec{v}) = \frac{\partial}{\partial x_i}(\vec{u}\times\vec{v})_i = \frac{\partial}{\partial x_i}\varepsilon_{ijk}u_j v_k$$

$$= \varepsilon_{ijk}\frac{\partial u_j}{\partial x_i}\cdot v_k + \varepsilon_{ijk}\frac{\partial v_k}{\partial x_i}u_j$$

$$= \varepsilon_{kij}\frac{\partial u_j}{\partial x_i}v_k - \varepsilon_{jik}\frac{\partial v_k}{\partial x_i}u_j$$

$$= [\vec{\nabla}\times\vec{u}]_k\cdot v_k - [\vec{\nabla}\times\vec{v}]_j\cdot u_j$$

$$= (\vec{\nabla}\times\vec{u})\cdot\vec{v} - (\vec{\nabla}\times\vec{v})\cdot\vec{u}$$

Definition Given a vector $\vec{u}$, define the operator $\vec{u}\cdot\vec{\nabla}$ to be $\vec{u}\cdot\vec{\nabla} = u_j \frac{\partial}{\partial x_j}$, which acts in the following way:

$$\vec{u}\cdot\vec{\nabla} f = u_j \frac{\partial f}{\partial x_j} = \vec{u}\cdot(\vec{\nabla} f)$$

$$\vec{u}\cdot\vec{\nabla}\,\vec{V} = u_j \frac{\partial V_i}{\partial x_j} = (\vec{u}\cdot(\vec{\nabla} V_1),\ \vec{u}\cdot(\vec{\nabla} V_2),\ \vec{u}\cdot(\vec{\nabla} V_3))$$

Example Show that $\vec{\nabla}(\vec{u}\cdot\vec{V}) = \vec{u}\times(\vec{\nabla}\times\vec{V}) + \vec{V}\times(\vec{\nabla}\times\vec{u}) + \vec{u}\cdot\vec{\nabla}\vec{V} + \vec{V}\cdot\vec{\nabla}\vec{u}$

$$[\vec{u}\times(\vec{\nabla}\times\vec{V})]_i = \varepsilon_{ijk} u_j (\vec{\nabla}\times\vec{V})_k = \varepsilon_{ijk} u_j \varepsilon_{klm} \frac{\partial V_m}{\partial x_l}$$

$$= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, u_j \frac{\partial V_m}{\partial x_l}$$

$$= \left(\delta_{il}\delta_{jm} u_j \frac{\partial V_m}{\partial x_l}\right) - \left(\delta_{im}\delta_{jl} u_j \frac{\partial V_m}{\partial x_l}\right)$$

$$= u_j \frac{\partial V_j}{\partial x_i} - u_j \frac{\partial V_i}{\partial x_j}$$

similarly, we can show

$$[\vec{V}\times(\vec{\nabla}\times\vec{u})]_i = V_j \frac{\partial u_j}{\partial x_i} - V_j \frac{\partial u_i}{\partial x_j}$$

So

$$[\vec{u}\times(\vec{\nabla}\times\vec{V})]_i + [\vec{V}\times(\vec{\nabla}\times\vec{u})]_i = u_j \frac{\partial V_j}{\partial x_i} - u_j \frac{\partial V_i}{\partial x_j} + V_j \frac{\partial u_j}{\partial x_i} - V_j \frac{\partial u_i}{\partial x_j}$$

$$= \left(u_j \frac{\partial V_j}{\partial x_i} + V_j \frac{\partial u_j}{\partial x_i}\right) - u_j \frac{\partial V_i}{\partial x_j} - V_j \frac{\partial u_i}{\partial x_j}$$

$$= \frac{\partial}{\partial x_i}(u_j V_j) - u_j \frac{\partial V_i}{\partial x_j} - V_j \frac{\partial u_i}{\partial x_j}$$

$$= [\vec{\nabla}(\vec{u}\cdot\vec{V})]_i - [\vec{u}\cdot\vec{\nabla}\vec{V}]_i - [\vec{V}\cdot\vec{\nabla}\vec{u}]_i$$