SUFFIX NOTATION

When we write the equations involving vectors, it often happen that we are doing the same thing on each of the coordinates. For example, if $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ then $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$.

It seems to be tedius to repeat the same thing 3 times. So we now introduce another way of representing vector operations, called the suffix notation.

If $\vec{u} = (u_1, u_2, u_3)$ is a vector, we represent it by u_i where it's understood that $i = 1, 2, 3$. Then the equation $\vec{u} + \vec{v} = \vec{w}$ can be written as

$$u_i + v_i = w_i .$$

The suffix "i" is called a free suffix. We say it's "free" because if we replace all the "i" by another letter, say "j", we refer to the same equation. But, you must make sure that the same letter to be used for each term of the expression.

Next we consider another type of operation, the dot product. We know if $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$ then

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{3} u_i v_i$$

So in suffix notation, we denote the dot product of $\vec{u}$ and $\vec{v}$ by:

$$u_i v_i$$

The repeated suffix means we should take the sum of the terms as i goes over from 1 to 3. We call it the Summation Convention: Whenever a suffix is repeated in a single term in an expression, we are to take the sum as the suffix goes from 1 to 3.
We call this kind of suffix a "dummy suffix".

- A principle is that any suffix should appear no more than twice in each term of an expression.

Now we can do more complicated examples: $(\vec{a}.\vec{b})(\vec{c}.\vec{d})$
We know $\vec{a}.\vec{b}$ is a scalar, and by suffix notation it can be expressed as $a_i b_i$; $\vec{c}.\vec{d}$ is also a scalar, and it can be expressed as $c_j d_j$. So we can write

$$(\vec{a}.\vec{b})(\vec{c}.\vec{d}) = a_i b_i c_j d_j$$

and it indeed means

$$(\vec{a}.\vec{b})(\vec{c}.\vec{d}) = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i b_i c_j d_j$$

Note we cannot choose the same suffix for $\vec{a}.\vec{b}$ and $\vec{c}.\vec{d}$, since each suffix can appear at most twice in each term. But we can reorder the letters:

$a_i c_j b_i d_j$ is the same as $a_i b_i c_j d_j$.

Example Write the suffix notation expression $a_j b_i c_j$ in ordinary vector notation

$$a_j b_i c_j = a_j c_j b_i$$

So the equation stands for the vector whose i-th component is $\sum_{j=1}^{3} a_j c_j b_i = (\vec{a}\cdot\vec{c}) b_i$. thus the equation is $(\vec{a}\cdot\vec{c})\vec{b}$

Example. Write the vector equation $\vec{u} + (\vec{a}\cdot\vec{b})\vec{v} = |\vec{a}|^2(\vec{b}\cdot\vec{v})\vec{a}$ in suffix notation

First, the i-th component of this vector equation

$$u_i + (\vec{a}\cdot\vec{b}) v_i = (\vec{a}\cdot\vec{a})(\vec{b}\cdot\vec{v}) a_i$$

Now replace each dot product by suffix notation, we get:

$$u_i + a_j b_j v_i = a_j a_j b_k v_k a_i$$

Example. We are going to show $tr(AB) = tr(BA)$ where A and B are two $n\times n$ matrices with real entries.

If $C = AB$, we know the ij-th entry of C is $C_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$, so by suffix notation, we can write

$$C_{ij} = A_{ik} B_{kj}$$

(Since k appear twice in the product $A_{ik} B_{kj}$, we take the sum of the terms $\sum_{k=1}^{n} A_{ik} B_{kj}$ by the rule)

Recall the trace of a matrix C is

$Tr(C) = \sum_{j=1}^{n} C_{jj}$. we can write in suffix notation as C_{jj}

The trace of AB is $C_{jj} = A_{jk} B_{kj}$.

Similarly, the trace of BA is $(BA)_{jj} = B_{jk} A_{kj}$

So we see $Tr(BA)$ is identified with $Tr(AB)$ by relabelling.

Definition. We define the Kronecker delta δ_{ij} to be the suffix notation of the $n \times n$ identity matrix. i.e.

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Lemma. $\delta_{ij} = \delta_{ji}$.

The δ_{ij} has lots of applications in computations with suffix notation.

Proposition (i) $\delta_{ij} a_i = a_j$

(ii) $\vec{a} \cdot \vec{b}$ can be written as $\delta_{ij} a_i b_j$ in suffix notation

Proof. (i) $\delta_{ij} a_i = \delta_{1j} a_1 + \delta_{2j} a_2 + \delta_{3j} a_3 = a_j$

(ii) $\delta_{ij} a_i b_j = \sum_{i=1}^{3} \sum_{j=1}^{3} \delta_{ij} a_i b_j = a_1 b_1 + a_2 b_2 + a_3 b_3 = \vec{a} \cdot \vec{b}$

Proposition $\delta_{ij}\delta_{jk} = \delta_{ik}$

Proof $\delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k}$

If $i \neq k$, then either $i \neq 1$ or $k \neq 1$, we get either $\delta_{i1} = 0$ or $\delta_{1k} = 0$

so $\delta_{i1}\delta_{1k} = 0$

similarly, $\delta_{i2}\delta_{2k} = 0$, $\delta_{i3}\delta_{3k} = 0$

we get in this case $\delta_{ij}\delta_{jk} = 0$

If $i = k$, $\delta_{ij}\delta_{jk} = \delta_{k1}\delta_{1k} + \delta_{k2}\delta_{2k} + \delta_{k3}\delta_{3k} = \delta_{kk}\cdot\delta_{kk} = 1$

We finish the proof.

Definition. The alternating tensor ε_{ijk} is defined to be

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if any of } i, j, k \text{ are equal} \\ +1 & \text{if } (i,j,k) = (1,2,3), (2,3,1) \text{ or } (3,1,2) \\ -1 & \text{if } (i,j,k) = (1,3,2), (3,2,1) \text{ or } (2,1,3) \end{cases}$$

Remark. When $\{i,j,k\} = \{1,2,3\}$, ε_{ijk} is indeed the signature (also called parity) of the permutation $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \in S_3$.

Proposition. (i) $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij}$

(ii) $\varepsilon_{ijk} = -\varepsilon_{jik}$

Proof Follow directly from definition

One application of the alternating tensor is to express cross product of vectors in suffix notation:

Proposition. $(\vec{a}\times\vec{b})_i = \varepsilon_{ijk} a_j b_k$

Proof We know $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \sum_{\sigma\in S_3} \operatorname{sgn}(\sigma)\, \vec{e}_{\sigma(1)}\, a_{\sigma(2)}\, b_{\sigma(3)}$

$$= \sum_{\substack{i,j,k \\ \text{distinct}}} \varepsilon_{ijk}\, a_j b_k \vec{e}_i$$

So the coefficient for $\vec{e}_i$ is $\sum_{j\neq i} \sum_{\substack{k\neq i \\ k\neq j}} \varepsilon_{ijk} a_j b_k = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk} a_j b_k$

since whenever two of i, j, k are the same $\varepsilon_{ijk} = 0$.

Example Write $\vec{a}\cdot(\vec{b}\times\vec{c})$ in suffix notation

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = a_i(\vec{b}\times\vec{c})_i = a_i \varepsilon_{ijk} b_j c_k = \varepsilon_{ijk} a_i b_j c_k$$

The above expression tells us:

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \varepsilon_{ijk} a_i b_j c_k = \varepsilon_{jki} b_j c_k a_i = \vec{b}\cdot(\vec{c}\times\vec{a})$$

Exercise What is $\varepsilon_{ijk}\varepsilon_{ijk}$?