CONSERVATIVE VECTOR FIELDS

Definition. A vector field $\vec{F}$ is conservative if it has the property that the line integral of $\vec{F}$ along any closed curve C is zero: $\oint_C \vec{F}\cdot d\vec{r} = 0$

Equivalently, a vector field $\vec{F}$ is conservative if the line integral of $\vec{F}$ along a curve only depends on the endpoints of the curve, i.e. independent of the path taken.

$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$ where C_1, C_2 are two curves that have the same endpoints.

Definition. Given a differentiable function $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$, we define the gradient of f to be the vector field $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

Theorem. $\vec{F}$ is a vector field on some region $D \subseteq \mathbb{R}^3$. Then $\vec{F}$ is conservative if and only if there exists a function $\phi: \mathbb{R}^3 \to \mathbb{R}$ such that $\vec{F} = \nabla\phi$

Proof. "$\Leftarrow$" If $\vec{F} = \nabla\phi$, then for any curve C from $A \in \mathbb{R}^3$ to $B \in \mathbb{R}^3$, parameterize C by $\vec{r}(t)$, $t \in [a,b]$, so that $\vec{r}(a) = A$, $\vec{r}(b) = B$. Write $\vec{r}(t) = (x(t), y(t), z(t))$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla\phi(\vec{r})\cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_a^b \frac{\partial\phi}{\partial x}(x(t),y(t),z(t))\frac{dx}{dt} + \frac{\partial\phi}{\partial y}(x(t),y(t),z(t))\frac{dy}{dt} + \frac{\partial\phi}{\partial z}(x(t),y(t),z(t))\, dt$$

$$= \int_a^b \frac{d\phi(\vec{r}(t))}{dt}\, dt$$

$$= \phi(\vec{r}(b)) - \phi(\vec{r}(a)) = \phi(B) - \phi(A)$$

so the integral only depends on the endpoints of C.

"$\Longrightarrow$". Conversely, if $\vec{F}$ is conservative on D,

define $\phi(x,y,z) = \int_C \vec{F} \cdot d\vec{r}$, where C is any path from $\vec{0}$ to (x,y,z)

We need to show $\nabla\phi = \vec{F} = (f_1, f_2, f_3)$

$$\frac{\partial \phi}{\partial x} = \lim_{h \to 0} \frac{\phi(x+h,y,z) - \phi(x,y,z)}{h}$$

$$= \lim_{h \to 0} \frac{\int_L \vec{F} \cdot d\vec{r}}{h}$$

where L is the line segment from (x,y,z) to $(x+h, y, z)$ parameterized by: $\vec{r}(t) = (t, y, z)$, $t \in [x, x+h]$

$$= \lim_{h \to 0} \frac{\int_x^{x+h} f_1(t,y,z)\,dt}{h}$$

$$= \lim_{h \to 0} \frac{\int_0^{x+h} f_1(t,y,z)\,dt - \int_0^x f_1(t,y,z)\,dt}{h}$$

$$= \frac{d}{dx} \int_0^x f_1(t,y,z)\,dt$$

$$= f_1(x,y,z)$$

Similarly, you can verify $\frac{\partial \phi}{\partial y} = f_2$ and $\frac{\partial \phi}{\partial z} = f_3$

Definition. If $\vec{F} = \nabla\phi$, we say ϕ is the potential for $\vec{F}$.

Example. $\vec{F}(x,y,z) = -\frac{MG}{|(x,y,z)|^3}(x,y,z)$ is conservative with potential function $\frac{MG}{\sqrt{x^2+y^2+z^2}}$

Example. $\vec{F}(x,y) = (y, -x)$ is not conservative on $\mathbb{R}^2$:

Suppose it's conservative, $\vec{F}(x,y) = \nabla\phi$ for some $\phi: \mathbb{R}^2 \to \mathbb{R}$

then $\begin{cases} \frac{\partial \phi}{\partial x} = y \\ \frac{\partial \phi}{\partial y} = -x \end{cases} \Rightarrow \begin{cases} \frac{\partial^2 \phi}{\partial y \partial x} = 1 \\ \frac{\partial^2 \phi}{\partial x \partial y} = -1 \end{cases}$ Contradicts to Young's Theorem

By the previous example, we see that for a 2-dimensional vector field $\vec{F}(x,y) = (f_1(x,y), f_2(x,y))$; a necessary condition for $\vec{F}$ to be conservative is that $\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x}$

We would like also to see whether this is a sufficient condition. And it turns out that we need to impose some conditions on the region.

Theorem. Let $\vec{F} = (P, Q)$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives, and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D, then $\vec{F}$ is conservative.

The proof of the above theorem requires the Green's Theorem. We will prove the Green's Theorem later as a special case of the Stokes Theorem, but we would like to state the Green's Theorem here:

Theorem (Green's Theorem). Let C be a positively oriented, piecewise smooth, simple closed curve in a plane, and D the region bounded by C. If $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives on an open region containing D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

In the previous discussion, a "simply connected" region means a region that has no holes.

If the vector field is defined on a region which is not simply connected, the condition $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ is not sufficient to conclude $\vec{F}$ is conservative.

Example. $\vec{F}(x,y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$

$$\frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}.$$

$$\text{so } \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)$$

But $\vec{F}(x,y)$ is not conservative:

Take C to be the unit circle. $x^2+y^2=1$.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\left(\frac{-\sin\theta}{\cos^2\theta+\sin^2\theta}, \frac{\cos\theta}{\cos^2\theta+\sin^2\theta}\right)\cdot(\cos\theta, \sin\theta)'\,d\theta$$

$$= \int_0^{2\pi}(-\sin\theta, \cos\theta)\cdot(-\sin\theta, \cos\theta)\,d\theta$$

$$= \int_0^{2\pi} \sin^2\theta + \cos^2\theta \, d\theta$$

$$= 2\pi \neq 0$$