VECTOR FIELD AND LINE INTEGRAL

The concept of vector field is widely used in physics, and it also has its own interest in mathematics.

Definition. A vector field on $\mathbb{R}^n$ is a function $\vec{F}$ that assigns to each $p \in \mathbb{R}^n$ an n-dimensional vector. If we write in coordinates, a vector field is a function

$$\vec{F}: \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$(x_1, \dots, x_n) \longmapsto (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n))$$

Example. We can define the Gravitational Field by

$$\vec{F}(x,y,z) = -\frac{MG}{|(x,y,z)|^3}(x,y,z)$$

Now we are going to study the integration of vector fields along some geometric objects. Intuitively, the integration tells us the net effect of some vector field on some geometric objects.

The first kind of integral we are interested is called the line integral.

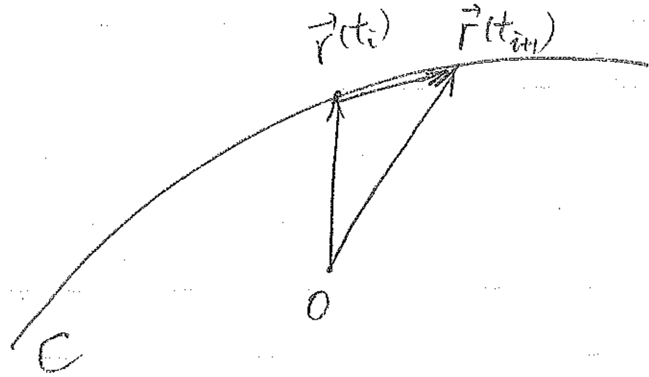
A curve in $\mathbb{R}^3$ can be described in coordinates by $\vec{r}(t) = (x(t), y(t), z(t))$

Definition. If $\vec{r}(t) = (x(t), y(t), z(t))$ is a curve, define the tangent vector of $\vec{r}(t)$ at $\vec{r}(t_0)$ to be $\vec{r}\,'(t_0) = (\frac{dx}{dt}(t_0), \frac{dy}{dt}(t_0), \frac{dz}{dt}(t_0))$

Given a vector field $\vec{F}(x,y,z)$ and a curve $C: \vec{r}(t)$, $t\in[a,b]$
we can define the line integral of $\vec{F}(x,y,z)$ along C
to be the limit of a riemann sum:

$$\int_C \vec{F}\cdot d\vec{r} = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{N} \vec{F}(\vec{r}(t_i))\cdot\Delta\vec{r} = \lim_{\max\Delta t_i\to 0} \sum_{i=1}^{N} \vec{F}(\vec{r}(t_i))\cdot(\vec{r}(t_i)-\vec{r}(t_{i-1}))$$

where $a=t_0<t_1<\cdots<t_{N-1}<t_N=b$.



The spirit of this definition is the observation that when $\vec{r}(t_i)$ and $\vec{r}(t_{i+1})$ are close to each other, $\vec{r}(t_{i+1})-\vec{r}(t_i)$ is a good approximation of the curve between $t=t_i$ and $t=t_{i+1}$

Similar to the integral of a function along the real line it's in general very hard to evaluate the integral by the Riemann sum. We need to develop some ways of integration:

$\frac{d\vec{r}}{dt} = \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}$, so when Δt is small,

$\vec{r}(t+\Delta t)-\vec{r}(t)$ can be approximated by $\frac{d\vec{r}}{dt}\Delta t$.

This indicates

$$\int_C \vec{F}\cdot d\vec{r} = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{N} \vec{F}(\vec{r}(t_i))\left(\vec{r}(t_i)-\vec{r}(t_{i-1})\right) = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{N} \vec{F}(\vec{r}(t_i))\cdot \frac{d\vec{r}}{dt}\Delta t_i$$

$$= \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}}{dt}\, dt$$

We thus obtain:

Proposition. $\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}}{dt}\, dt$

Example. $\vec{F}(x,y,z) = (y, x, z)$. the curve C is parameterized by $\vec{r}(t) = (t, t^2, 2t^2)$, $t \in [0,1]$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}}{dt}\, dt$$

$$= \int_0^1 (t^2, t, 2t^2)\cdot(1, 2t, 4t)\, dt$$

$$= \int_0^1 t^2 + 2t^2 + 8t^3\, dt$$

$$= 3$$

Definition. If C is a closed path, i.e. when the starting and end points coincide, the line integral of $\vec{F}$ along C is written as $\oint_C \vec{F}\cdot d\vec{r}$, and we call it the circulation of $\vec{F}$ around C.

Proposition. If we change the orientation of C, the line integral is:

$$\int_{-C} \vec{F}\cdot d\vec{r} = -\int_C \vec{F}\cdot d\vec{r}$$

where $-C$ is the curve obtained by reversing the orientation of C.

Proof. It follows directly from the definition, since the tangent vectors of C and $-C$ at each point are opposite to each other.

There is one more thing we need to take care of:
A geometric curve C may have more than one parameterizations, so we need to make sure different choices of parameterizations lead to the same line integral.

Suppose C can also be parameterized by s with $t=f(s)$ for some function f, i.e. $\vec{\rho}(s)=\vec{r}(f(s))$. Then

$$\int_a^b \vec{F}(\vec{\rho}(s)).\frac{d\vec{\rho}}{ds}\cdot ds=\int_a^b \vec{F}(\vec{r}(f(s))).\frac{d\vec{r}}{dt}\cdot f'(s)ds$$

$$=\int_{f(a)}^{f(b)} \vec{F}(\vec{r}(t)).\frac{d\vec{r}}{dt}\,dt$$

Proposition. The line integral is independent of the parameterization of the path C.

There're also other forms of line integrals: if we consider the scalar multiplication and cross product instead of dot product.

If $f:\mathbb{R}^3\to\mathbb{R}$ is a scalar function, C is a curve parameterized by $\vec{r}(t)$, $t\in[a,b]$. define the line integral

$$\int_C f\,d\vec{r}=\int_a^b f\,\frac{d\vec{r}}{dt}\,dt=\left(\int_a^b f\,\frac{dx}{dt}\,dt,\ \int_a^b f\,\frac{dy}{dt}\,dt,\ \int_a^b f\,\frac{dz}{dt}\,dt\right)$$