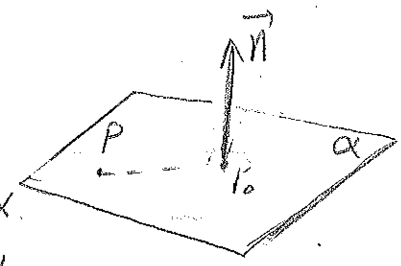
An important application of dot product is in geometry. We can describe a plane in $\mathbb{R}^3$ by an equation that involves dot product:



Pick a vector $\vec{n} = (a, b, c)$ perpendicular to the plane α.
Pick a point $P_0 = (x_0, y_0, z_0) \in \alpha$.
Now for any $P = (x, y, z)$, $P \in \alpha$ if and only if $\overrightarrow{P_0P} \perp \vec{n}$. (We call $\vec{n}$ a normal vector of α).

Algebraically, $\overrightarrow{P_0P} \perp \vec{n}$ is equivalent to
$$(x-x_0, y-y_0, z-z_0) \cdot (a, b, c) = 0.$$

i.e. $a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

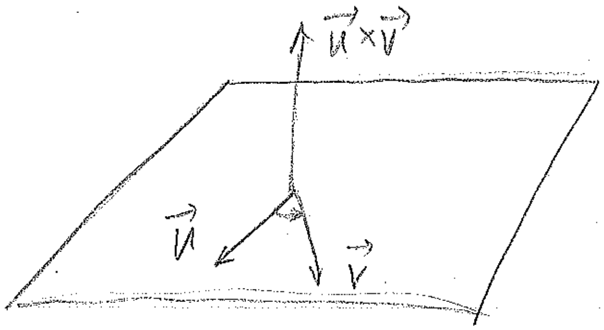
or $\quad ax + by + cz = ax_0 + by_0 + cz_0$

Example: The plane that passes through $(1, 2, 3)$ with a normal vector $(5, -2, 1)$ has equation
$$5(x-1) - 2(x-2) + (x-1) = 0.$$

The dot product assigns a real number (i.e. scalar) to a given pair of vectors, as we've seen in its definition. Now we're going to consider another form of vector product, called cross product.

Definition. $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^3$. Define the cross product $\vec{u}\times\vec{v}$ to be the vector whose magnitude is $|\vec{u}|\cdot|\vec{v}|\sin\theta$, whose direction is determined by the "right hand rule".
(θ is the angle between $\vec{u}$ and $\vec{v}$)

Example.



$\vec{u}\times\vec{v}$ is a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

Proposition. $\vec{u}$, $\vec{v}$, $\vec{w}$ are vectors in $\mathbb{R}^3$

(i) $\vec{u}\times\vec{v} = -\vec{v}\times\vec{u}$

(ii) $\vec{u}\times\vec{0} = \vec{0}$

(iii) $\vec{u}\times\vec{u} = \vec{0}$

(iv) $\vec{u}\,/\!/\,\vec{v} \Rightarrow \vec{u}\times\vec{v} = \vec{0}$

(v) $|\vec{u}\times\vec{v}|$ equals to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$

(vi) $\vec{u}\times(\vec{v}+\vec{w}) = \vec{u}\times\vec{v}+\vec{u}\times\vec{w}$

Proof. (i) By the right hand rule, if the order of the two vectors is switched, the direction of the cross product will be changed to the opposite.

(ii). $|\vec{u}\times\vec{0}| = |\vec{u}|\cdot|\vec{0}|\cdot\sin\theta = 0$, so $\vec{u}\times\vec{0} = \vec{0}$

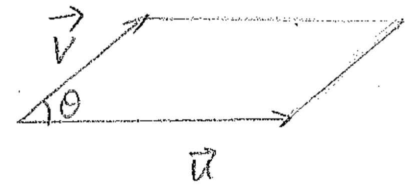
(iii). by (i), let $\vec{v}=\vec{u}$, then $\vec{u}\times\vec{u} = -\vec{u}\times\vec{u}$, so $\vec{u}\times\vec{u} = \vec{0}$

(iv). If $\vec{u} \parallel \vec{v}$, then $\theta = 0$ or $\theta = \pi$, $\sin 0 = \sin \pi = 0$

$$|\vec{u} \times \vec{v}| = |\vec{u}| \cdot |\vec{v}| \sin\theta = 0.$$

(v). By the Sine Theorem, the area of the parallelogram is

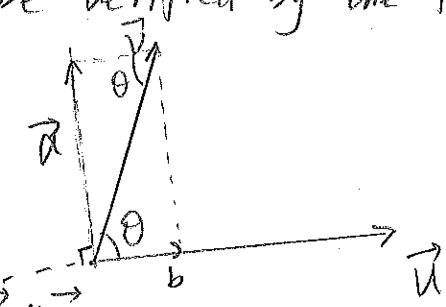
$$2 \cdot \frac{|\vec{u}| \cdot |\vec{v}| \sin\theta}{2} = |\vec{u} \times \vec{v}|$$



(vi). To prove (vi), we need a Lemma:

Lemma. The cross product of $\vec{u}$ and $\vec{v}$ only depends on the Component of $\vec{u}$ perpendicular to $\vec{u}$.

This Lemma can be verified by the following picture:



we decompose $\vec{v} = \vec{a} + \vec{b}$ such that $\vec{a} \perp \vec{u}$ and $\vec{b} \parallel \vec{u}$.

Then $\vec{u} \times \vec{v} = \vec{u} \times \vec{a}$.

Now with the help of this Lemma, we can prove (vi). $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$:

Let $\vec{v} = \vec{a} + \vec{b}$, $\vec{w} = \vec{c} + \vec{d}$, such that

$\vec{a} \perp \vec{u}$, $\vec{b} \parallel \vec{u}$, $\vec{c} \perp \vec{u}$, $\vec{d} \parallel \vec{u}$.

by the Lemma, we know that

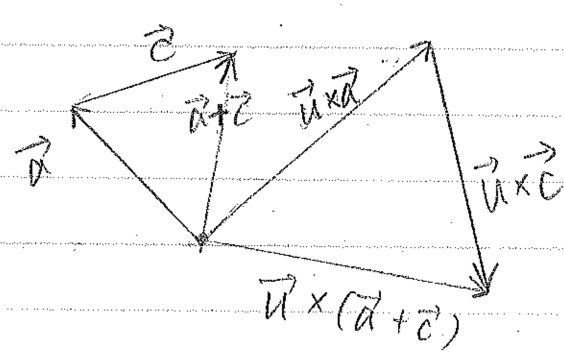
$$\vec{u}\times(\vec{v}+\vec{w})=\vec{u}\times(\vec{a}+\vec{b}+\vec{c}+\vec{d})=\vec{u}\times((\vec{a}+\vec{c})+(\vec{b}+\vec{d}))$$
$$=\vec{u}\times(\vec{a}+\vec{c})$$

$$\vec{u}\times\vec{v}+\vec{u}\times\vec{w}=\vec{u}\times\vec{a}+\vec{u}\times\vec{c}$$

So it suffices to verify $\vec{u}\times(\vec{a}+\vec{c})=\vec{u}\times\vec{a}+\vec{u}\times\vec{c}$ for $\vec{a}\perp\vec{u}$, $\vec{c}\perp\vec{u}$. This can be done as follows:

Consider $\vec{u}$ to be pointing into the page, so $\vec{a}$ and $\vec{c}$ are parallel to the page.

$\vec{u}\times\vec{a}$ is the vector of magnitude $|\vec{u}|\cdot|\vec{a}|$, in the direction of $\vec{a}$ rotating clockwise $90°$,
$\vec{u}\times\vec{c}$ is the vector of magnitude $|\vec{u}|\cdot|\vec{b}|$, in the direction of $\vec{c}$ rotating clockwise $90°$.
$\vec{u}\times(\vec{a}+\vec{c})$ is the vector of magnitude $|\vec{u}|\cdot|\vec{a}+\vec{c}|$, in the direction of $\vec{a}+\vec{c}$ rotating clockwise $90°$.
Then geometrically we see very clearly that $\vec{u}\times(\vec{a}+\vec{c})$ coincides with $\vec{u}\times\vec{a}+\vec{u}\times\vec{c}$



Now with the help of the distributive law, we can develop an algebraic way of computation.

Again we take the standard basis $\{\vec{i}, \vec{j}, \vec{k}\}$.
Observe that $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$.

If $\vec{u}=(x_1, y_1, z_1)$, $\vec{v}=(x_2, y_2, z_2)$, then
$\vec{u}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}$, $\vec{v}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$.

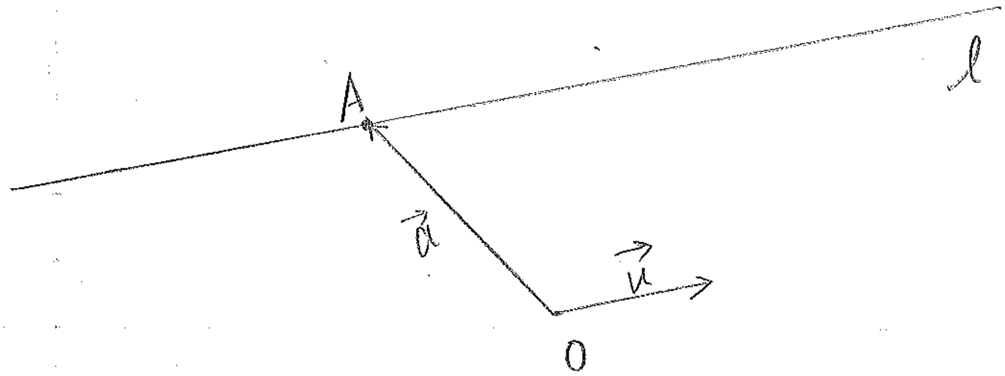
$$
\begin{aligned}
\vec{u}\times\vec{v} &= (x_1\vec{i}+y_1\vec{j}+z_1\vec{k})\times(x_2\vec{i}+y_2\vec{j}+z_2\vec{k}) \\
&= (x_1\vec{i})\times(y_2\vec{j})+(x_1\vec{i})\times(z_2\vec{k})+(y_1\vec{j})\times(x_2\vec{i}) \\
&\quad +(y_1\vec{j})\times(z_2\vec{k})+(z_1\vec{k})\times(x_2\vec{i})+(z_1\vec{k})\times(y_2\vec{j}) \\
&= (y_1z_2-z_1y_2)\vec{i}+(z_1x_2-x_1z_2)\vec{j}+(x_1y_2-y_1x_2)\vec{k}
\end{aligned}
$$

An interesting observation is that above indicates:

$$
\vec{u}\times\vec{v}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}
$$

The cross product also has applications in geometry: it can be used to describe straight lines in R^3.

Given a straight line $l \subseteq \mathbb{R}^3$, it can be described as $\vec{r}=\vec{a}+\lambda\vec{u}$, where $\vec{a}$ is the position vector of a Point $A=(a_1, a_2, a_3)$ on l, and $\vec{u} \parallel l$.
λ is the parameter.



We wish to obtain a form of equation without the parameter λ, so we use the cross product:

$$\vec{r}\times\vec{u} = (\vec{a}+\lambda\vec{u})\times\vec{u} = \vec{a}\times\vec{u}$$

i.e. $\vec{r}\times\vec{u} = \vec{a}\times\vec{u}$

Note $\vec{a}$ and $\vec{u}$ are some fixed vectors, It follows $\vec{a}\times\vec{u} = \vec{b}$ is some fixed vector.

So the equation of l can be written as

$$\vec{r}\times\vec{u} = \vec{b}.$$

Now we are going to see another product of vectors: Scalar Triple Product.

Definition. Given three vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$ their scalar triple product is $\vec{u}\cdot(\vec{v}\times\vec{w})$

If $\vec{u} = (u_1, u_2, u_3)$, $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$ then

$$\vec{u}\cdot(\vec{v}\times\vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Proposition. (i) $\vec{u}\cdot(\vec{v}\times\vec{w}) = \vec{v}\cdot(\vec{w}\times\vec{u}) = \vec{w}\cdot(\vec{u}\times\vec{v})$

(ii) $\vec{u}\cdot(\vec{v}\times\vec{w}) = (\vec{u}\times\vec{v})\cdot\vec{w}$.

(iii) $\vec{u}\cdot(\vec{v}\times\vec{w}) \neq 0$ if and only if $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

(iv) $\vec{u}\cdot(\vec{v}\times\vec{w})$ is the signed volume of the parallelepiped formed by $\vec{a}, \vec{b}, \vec{c}$.

Proof. The proofs follow easily from elementary properties of determinant.