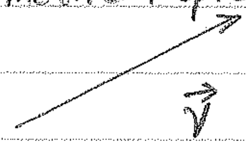
VECTORS IN $\mathbb{R}^N$.

Definition. A (real) vector in $\mathbb{R}^N$ is a quantity which has both magnitude and direction. The magnitude is a nonnegative number, and the directions are described by rays in $\mathbb{R}^N$

Geometric Representation of Vectors:

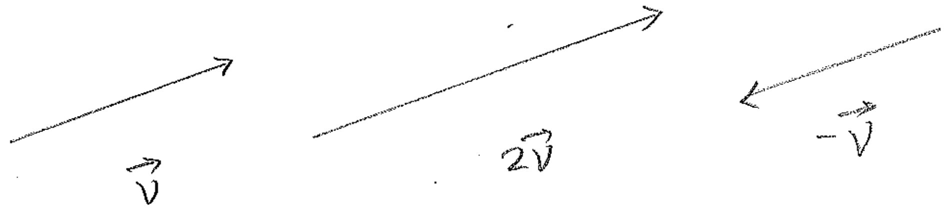


An oriented line segment in $\mathbb{R}^N$ represents a vector $\vec{v}$. The length of the line segments represents the magnitude of $\vec{v}$ and the orientation represents the direction of $\vec{v}$.

Note that a vector is determined by its magnitude and direction, but not its position. In other words, if we can translate a vector to coincide with another, then these two vectors are equal.

Now we are going to review the basic algebraic operations on vectors.

- Scalar Multiplication: If $\vec{v}$ is a vector and c is a real number (called a scalar), we can define the scalar multiplication $c\vec{v}$, which is a vector, as follows:
  The magnitude of $c\vec{v}$ is $|c|$ times the magnitude of $\vec{v}$, and for the direction:
  ① If $c>0$, $c\vec{v}$ has the same direction as that of $\vec{v}$
  ② If $c=0$, $c\vec{v}=\vec{0}$, no direction
  ③ If $c<0$, $c\vec{v}$ has the opposite direction as that of $\vec{v}$



And we write $-\vec{v}$ for $(-1)\vec{v}$

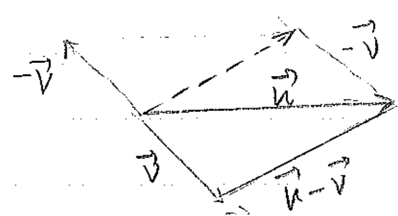
- Vector Addition:

If $\vec{u}$ and $\vec{v}$ are vectors positioned so the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then $\vec{u}+\vec{v}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.

Note that vector addition is commutative: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$
and associative: $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$.
There's also a distributive law of scalars on vectors:
$$c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$$
The above laws can be verified using some geometry.

- Vector Subtraction

If $\vec{u}$ and $\vec{v}$ are vectors, then define $\vec{u}-\vec{v}$ to be $\vec{u}+(-\vec{v})$



- Vector Representation in Coordinates: (We'll take $\mathbb{R}^3$ as example, but the principle extends naturally to all $\mathbb{R}^N$)

In a coordinate system, for any point $P=(x,y,z)$, we can construct the vector with initial point the origin $O=(0,0,0)$, and terminal point P. We denote this vector $\overrightarrow{OP}$ and call it the position vector of the point P

Observe that the assignment of position vector to point P gives an identification of (3-dimensional) vectors and points in $\mathbb{R}^3$. Another observation is that for each vector $\vec{v}$ and any point $A=(a_1,a_2,a_3)\in\mathbb{R}^3$, there is a point $B=(b_1,b_2,b_3)$ such that $\vec{v}$ is equivalent to $\overrightarrow{AB}$. We say $\overrightarrow{AB}$ is a representation of the vector $\vec{v}$.

In a coordinate system, if $\vec{v}$ is equivalent to the position vector of $P=(a,b,c)$, we can write $\vec{v}=(a,b,c)$

The coordinate system makes vector algebra simple to compute:

Theorem. If $\vec{u}=(a_1,b_1,c_1)$, $\vec{v}=(a_2,b_2,c_2)$ and $\lambda\in\mathbb{R}$, then:

$$\vec{u}+\vec{v}=(a_1+a_2,\ b_1+b_2,\ c_1+c_2),\quad \vec{u}-\vec{v}=(a_1-a_2,\ b_1-b_2,\ c_1-c_2)$$

$$\lambda\vec{u}=(\lambda a_1,\ \lambda b_1,\ \lambda c_1)$$

Theorem: If $\vec{u}=(a,b,c)$, then $|\vec{u}|=\sqrt{a^2+b^2+c^2}$

Example. If $\vec{u}=(3,2,5)$, $\vec{v}=(4,1,3)$, then

$$\vec{u}-2\vec{v}=(3,2,5)-2\cdot(4,1,3)=(3,2,5)-(8,2,6)$$

$$=(-5,0,-1)$$

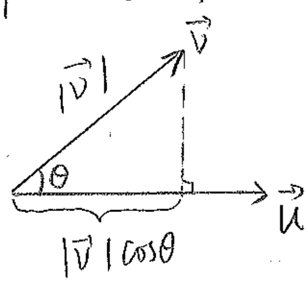
Definition. If a vector has length 1, we call it a unit vector.

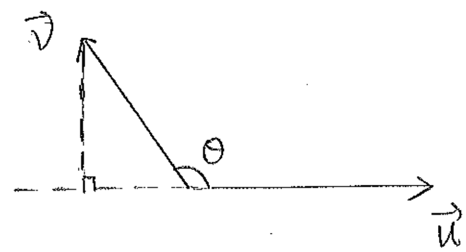
For example, $\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$, $\vec{k}=(0,0,1)$

Observe that if $\vec{v}$ is a nonzero vector, then there's a unit vector which has the same direction as $\vec{v}$, and the unit vector is $\frac{1}{|\vec{v}|}\vec{v}$.

Example $\vec{v}=(2,-2,-1)$. $|\vec{v}|=\sqrt{2^2+(-2)^2+(-1)^2}=3$, so the unit vector in the direction of $\vec{v}$ is $\left(\frac{2}{3},-\frac{2}{3},-\frac{1}{3}\right)$.

Definition. If θ is the angle between two vectors $\vec{u}$ and $\vec{v}$, we define the dot product of $\vec{u}$ and $\vec{v}$ to be $\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta$





By this definition, we can immediately see some of the properties of dot product:

(i) $\vec{u}\cdot\vec{v}=\vec{v}\cdot\vec{u}\quad\forall\,\vec{u},\vec{v}$.

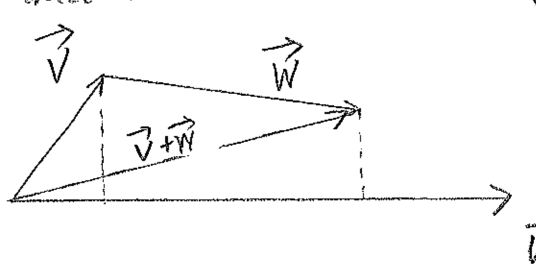
(ii) $\vec{u}\cdot\vec{0}=0\quad\forall\,\vec{u}$.

(iii) If $\theta=\frac{\pi}{2}$, i.e. $\vec{u}\perp\vec{v}$, then $\vec{u}\cdot\vec{v}=0$

(iv) $c(\vec{u}\cdot\vec{v})=(c\vec{u})\cdot\vec{v}=\vec{u}\cdot(c\vec{v})\quad\forall\,\vec{u},\vec{v},\ \forall\,c\in\mathbb{R}$

Lemma. Dot Product is distributive: $\vec{u}\cdot(\vec{v}+\vec{w})=\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

Proof.



From Linear Algebra, we know that we can choose a standard basis of $\mathbb{R}^3$, $\{\vec{i}, \vec{j}, \vec{k}\}$ such that $|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$,

they're pairwisely perpendicular to each other, and they're the unit vectors in the direction of the Cartisian Coordinate axis. i.e in coordinates, $\vec{i} = (1, 0, 0)$, $\vec{j} = (0, 1, 0)$ and $\vec{k} = (0, 0, 1)$

Each vector $\vec{u} = (a, b, c)$ can therefore be decomposed into
$$\begin{aligned}\vec{u} &= (a, 0, 0) + (0, b, 0) + (0, 0, c)\\ &= a(1,0,0) + b(0,1,0) + c(0,0,1)\\ &= a\vec{i} + b\vec{j} + c\vec{k}\end{aligned}$$

Lemma If $\vec{u} = (a_1, b_1, c_1)$, $\vec{v} = (a_2, b_2, c_2)$, then $\vec{u} \cdot \vec{v} = a_1a_2 + b_1b_2 + c_1c_2$

Proof
$$\begin{aligned}\vec{u} \cdot \vec{v} &= (a_1\vec{i} + b_1\vec{j} + c_1\vec{k}) \cdot (a_2\vec{i} + b_2\vec{j} + c_2\vec{k})\\ &= (a_1\vec{i} \cdot a_2\vec{i}) + (a_1\vec{i} \cdot b_2\vec{j}) + (a_1\vec{i} \cdot c_2\vec{k})\\ &\quad + (b_1\vec{j} \cdot a_2\vec{i}) + (b_1\vec{j} \cdot b_2\vec{j}) + (b_1\vec{j} \cdot c_2\vec{k})\\ &\quad + (c_1\vec{k} \cdot a_2\vec{i}) + (c_1\vec{k} \cdot b_2\vec{j}) + (c_1\vec{k} \cdot c_2\vec{k})\\ &= a_1a_2 + b_1b_2 + c_1c_2\end{aligned}$$
(since $\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$)

Lemma. $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$

Proof. $\vec{v} \cdot \vec{v} = |\vec{v}||\vec{v}| \cos 0 = |\vec{v}|^2 \cdot 1 = |\vec{v}|^2$

Corollary If $\vec{v} = (a, b, c)$, then $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$