DIVERGENCE

Definition $\vec{F}(x,y,z)$ is a vector field defined in a neighbourhood of $(x_0, y_0, z_0) \in \mathbb{R}^3$. V is a solid with (x_0, y_0, z_0) in its interior, $\partial V = \vec{S}$ is outward oriented. Define the divergence of $\vec{F}(x,y,z)$ at (x_0, y_0, z_0) to be:

$$\operatorname{div}\vec{F}(x_0, y_0, z_0) = \lim_{V \to 0} \frac{1}{V} \oiint_S \vec{F} \cdot d\vec{S}$$

So as (x_0, y_0, z_0) varies around, we obtain a scalar function $\operatorname{div}\vec{F}$.

Proposition (i) If λ and μ are constants, $\vec{F}$ and $\vec{G}$ are vector fields, then

$$\operatorname{div}(\lambda\vec{F} + \mu\vec{G}) = \lambda \operatorname{div}(\vec{F}) + \mu \operatorname{div}(\vec{G})$$

(ii) If f is a scalar function and $\vec{u}$ is a constant vector, then

$$\operatorname{div}(f\vec{u}) = \vec{u} \cdot \nabla f$$

Proof. (i) Follows directly from the distributive law of dot product

(ii)
$$\operatorname{div}(f\vec{u}) = \lim_{V \to 0} \frac{1}{V} \oiint_S (f\vec{u}) \cdot d\vec{S} = \lim_{V \to 0} \frac{1}{V} \vec{u} \cdot \oiint_S f \, d\vec{S}$$

$$= \vec{u} \cdot \lim_{V \to 0} \frac{1}{V} \oiint_S f \, d\vec{S}$$

$$= \vec{u} \cdot \nabla f$$

Proposition. If $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is a vector field, then $\operatorname{div}\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

So formally we write $\operatorname{div}\vec{F} = \vec{\nabla}.\vec{F}$

Proof $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

$$\begin{aligned}
\operatorname{div}\vec{F} &= \operatorname{div}(P\hat{i} + Q\hat{j} + R\hat{k}) \\
&= \operatorname{div}(P\hat{i}) + \operatorname{div}(Q\hat{j}) + \operatorname{div}(R\hat{k}) \\
&= \nabla P.\hat{i} + \nabla Q.\hat{j} + \nabla R.\hat{k} \\
&= \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}\right).(1,0,0) + \left(\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}\right).(0,1,0) \\
&\quad + \left(\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \frac{\partial R}{\partial z}\right).(0,0,1) \\
&= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}
\end{aligned}$$

Proposition. $\operatorname{div}(\operatorname{curl}\vec{F}) = 0$

Proof. $\operatorname{div}(\operatorname{curl}\vec{F}) = \operatorname{div}(R_y - Q_z, P_z - R_x, Q_x - P_y)$

$$\begin{aligned}
&= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} \\
&= 0
\end{aligned}$$

Definition. If f is a scalar function, define the Laplacian of f to be $\Delta f = \vec{\nabla}.(\vec{\nabla} f) = \operatorname{div}(\nabla f)$

$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Exercise. Show that $\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f.\nabla g$

GAUSS' THEOREM (DIVERGENCE THEOREM)

Theorem. V is a solid with boundary $\partial V = \vec{S}$ outward oriented.
$\vec{F}$ is a vector field defined on a region containing V.
Then:

$$\iiint_V \operatorname{div}\vec{F}\, dV = \oint\!\!\oint_S \vec{F}\cdot d\vec{S}$$

Proof. By the definition of $\operatorname{div}\vec{F} = \lim_{V\to 0} \frac{1}{V} \oint\!\!\oint_S \vec{F}\cdot d\vec{S}$

For each (x_0, y_0, z_0), $\operatorname{div}\vec{F}(x_0, y_0, z_0) = \frac{1}{V}\oint\!\!\oint_S \vec{F}\cdot d\vec{S} + \varepsilon(V)$

and $\varepsilon(V) \longrightarrow 0$ as $V \longrightarrow 0$.

so $\qquad \operatorname{div}\vec{F}(x,y,z)\, V = \oint\!\!\oint_S \vec{F}\cdot d\vec{S} + V\cdot\varepsilon(V)$

$$\iiint_V \operatorname{div}\vec{F}\, dV = \lim_{\max \Delta V_i \to 0} \sum_{\Delta V_i} \operatorname{div}\vec{F}(x_i^*, y_i^*, z_i^*)\,\Delta V_i$$

$$= \lim_{\max \Delta V_i \to 0} \sum_{\Delta V_i} \oint\!\!\oint_{\Delta V_i} \vec{F}\cdot d\vec{S} + \Delta V_i \cdot \varepsilon(\Delta V_i)$$

$$\overset{(*)}{=} \sum_{\Delta V_i} \oint\!\!\oint_{\Delta V_i} \vec{F}\cdot d\vec{S}$$

$$= \oint\!\!\oint_S \vec{F}\cdot d\vec{S}$$

$(*)$ is because $\left| \sum_{\Delta V_i} \Delta V_i \cdot \varepsilon(\Delta V_i) \right| \le \left| \max \varepsilon(\Delta V_i) \right| \sum_{\Delta V_i} \Delta V_i$

$$= \left| \max \varepsilon(\Delta V_i) \right| \cdot V \longrightarrow 0$$

Corollary. $\iiint_V \nabla f \, dV = \oiint_{\partial V} f \, d\vec{S}$. for a solid V.

Proof. For any constant vector $\vec{u}$, apply Gauss' Theorem to $f\vec{u}$.

$$\iiint_V \operatorname{div}(f\vec{u}) \, dV = \oiint_{\partial V} (f\vec{u}) . d\vec{S}$$

$$\iiint_V \vec{u} . \nabla f \, dV = \iint_{\partial V} \vec{u} . (f\hat{n}) \, dS$$

$$\vec{u} . \iiint_V \nabla f \, dV = \vec{u} . \oiint_{\partial V} f \, d\vec{S}$$

Since this holds for all $\vec{u} \in \mathbb{R}^3$ we conclude

$$\iiint_V \nabla f \, dV = \oiint_{\partial V} f \, d\vec{S}$$

Corollary. If $\vec{S}$ is a closed surface. $\oiint_S 1 \, d\vec{S} = \vec{0}$

Corollary. If $\vec{S}$ is a closed surface. $\iint_S \operatorname{curl} \vec{F} . d\vec{S} = 0$ for any vector field $\vec{F}$.

Proof. $\oiint_S \operatorname{curl} \vec{F} . d\vec{S} = \iiint_V \operatorname{div}(\operatorname{curl} \vec{F}) \, dV = \iiint_V 0 \, dV = 0$

Example. Evaluate $\oiint_S (3x, 2y, z) \cdot d\vec{S}$ where $\vec{S}$ is the unit sphere with centre $(0,0,0)$.

$$\oiint_S (3x, 2y, z) \cdot d\vec{S} = \iiint_B \frac{\partial(3x)}{\partial x} + \frac{\partial(2y)}{\partial y} + \frac{\partial(z)}{\partial z}\, dV$$

$$= \iiint_B 6\, dV$$

$$= 6 \iiint_B 1\, dV$$

$$= 6 \cdot \frac{4}{3}\pi$$

$$= 8\pi$$

Remark. The Green's Theorem can also be viewed as a special case of Gauss' Theorem:

If we parameterize a simple closed curve C on $\mathbb{R}^2$ by $\vec{r}(t) = (x(t), y(t))$, then let $\hat{n}(t) = \frac{1}{|\vec{r}'(t)|} \cdot (y'(t), -x'(t))$, $\hat{n}(t)$ is the unit normal vector of C at $\vec{r}(t)$ pointing outward.

$S = s(t) = \int_a^t |\vec{r}'(\tau)| d\tau$ is the arclength, then the Green's Theorem implies:

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \hat{n}(t) |\vec{r}'(t)| dt$$

$$= \int_a^b \frac{1}{|\vec{r}'(t)|} \cdot (P(\vec{r}(t))y'(t) - Q(\vec{r}(t))x'(t)) \cdot |\vec{r}'(t)| dt$$

$$= \oint_C P\, dy - Q\, dx$$

$$= \oint_C (-Q, P) \cdot d\vec{r} = \iint_D P_x + Q_y\, dA = \iint_D \operatorname{div}\vec{F}\, dA$$