Review of Triple Integral

We will first have a brief review of triple integral before discussing about divergence.
Recall that if E is some solid in $\mathbb{R}^3$ and f is a function defined on E, we define the triple integral of f along E to be the Riemann Sum

$$\iiint_E f\,dV = \lim_{\max \Delta V_i \to 0} \sum_{\Delta V_i} f(x_i^*, y_i^*, z_i^*)\Delta V_i$$

A good way of understanding triple integral is to regard f as the density function, then $\iiint_E f\,dV$ gives the mass of E

The evaluation of $\iiint_E f\,dV$ is in general complicated, but in some special cases we can translate it into the iterated form. For example, if E is of the form

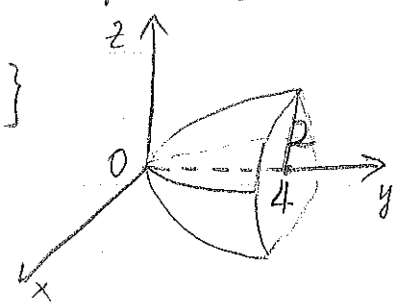
$$E = \{(x,y,z) \in \mathbb{R}^3 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x,y) \le z \le u_2(x,y)\}$$

then $\displaystyle\iiint_E f\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA$

where $D = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

Example. Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$ where E is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y=4$.

$E = \{(x,y,z) \in \mathbb{R}^3 \mid (x,z) \in D,\ x^2+z^2 \le y \le 4\}$
where $D = \{(x,z) \in \mathbb{R}^2 \mid x^2+z^2 \le 4\}$



So the integral is

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_D \left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right) dA$$

$$= \iint_D \sqrt{x^2+z^2}\,(4-x^2-z^2)\, dA$$

$$= \int_0^{2\pi}\int_0^2 r\,(4-r^2)\, r\, dr\, d\theta$$

$$= \frac{128}{15}\pi$$

Example Compute the volume of a ball of radius r.

We may take the ball $x^2+y^2+z^2=r^2$, call it B.

Then the volume is:

$$\iiint_B 1\, dV = \int_0^r \int_0^\pi \int_0^{2\pi} \rho^2 \sin\phi\, d\theta\, d\phi\, d\rho$$

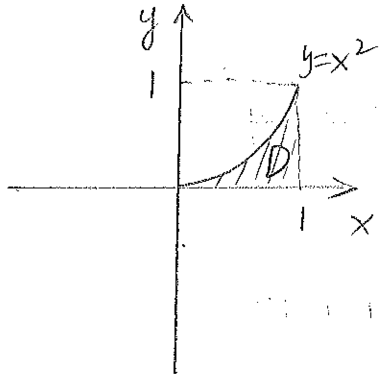
$$= \int_0^r \rho^2\, d\rho \cdot \int_0^\pi \sin\phi\, d\phi \cdot \int_0^{2\pi} 1\, d\theta$$

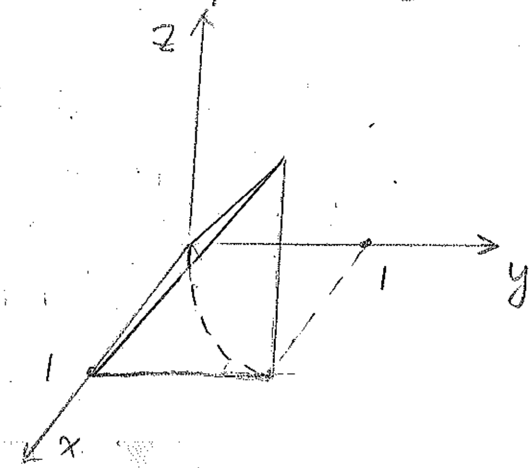
$$= \frac{r^3}{3} \cdot 2 \cdot 2\pi$$

$$= \frac{4}{3}\pi r^3$$

Example Rewrite $\int_0^1\int_0^{x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx$ into the form $\iiint f(x,y,z)\,dx\,dz\,dy$

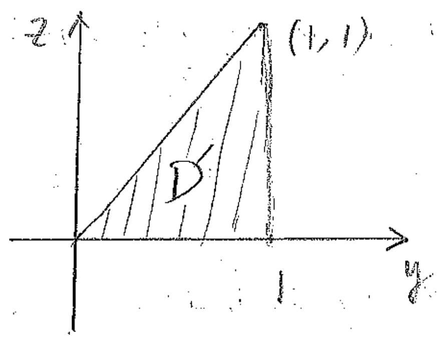
First, we need to find the solid represented by the integral





$E=\{(x,y,z)\in\mathbb{R}^3 \mid 0\le x\le 1,\ 0\le y\le x^2,\ 0\le z\le y\}$

The projection of E on y-z plane is



$D'=\{(y,z)\in\mathbb{R}^2 \mid 0\le y\le 1,\ 0\le z\le y\}$

$E=\{(x,y,z)\in\mathbb{R}^3 \mid (y,z)\in D',\ \sqrt{y}\le x\le 1\}$

So $\iiint_E f\,dV=\int_0^1\int_0^y\int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$