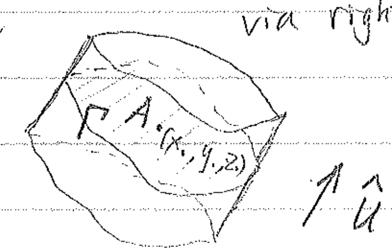
Next we are going to find another interpretation of curl, which will be useful in the proof of Stokes' Theorem.

Theorem: If $\vec{F}$ is a vector field, $\hat{u}$ is a constant unit vector, A is a flat surface with boundary curve Γ, and (x_0, y_0, z_0) is a point in the interior of A and $\hat{u} \perp A$. Then:

$$\hat{u} \cdot \operatorname{Curl} \vec{F}(x_0, y_0, z_0) = \lim_{A \to 0} \frac{1}{A} \oint_{\Gamma} \vec{F} \cdot d\vec{r}$$

where the orientation of Γ agrees with $\hat{u}$ via right-hand rule

We will here only provide a sketched proof. The reader may refer to Barry Spain's Vector Analysis book to fill in more technical details



Proof: We thicken A to be a solid of thickness $2l$, with A the middle layer. We call this solid $A \times I$. (I means the interval $[-l, l]$)

$$\partial(A \times I) = \partial A \times I \cup A \times \partial I$$
$$= \underbrace{\Gamma \times I}_{\text{side}} \cup \underbrace{A \times \{-l\}}_{\text{bottom}} \cup \underbrace{A \times \{l\}}_{\text{top}}$$

$$\hat{u} \cdot \operatorname{Curl} \vec{F}(x_0, y_0, z_0) = \hat{u} \cdot \left(-\lim_{A \times I \to 0} \frac{1}{A \times I} \oiint_{\partial(A \times I)} \vec{F} \times d\vec{S} \right)$$

$$= \lim_{A \times I \to 0} \frac{1}{A \times I} \oiint_{\partial(A \times I)} \vec{F} \cdot (\hat{u} \times \hat{n})\, dS$$

$$= \lim_{A \times I \to 0} \frac{1}{A \times I} \oiint_{\Gamma \times I} \vec{F} \cdot \hat{T}\, dS$$

let $\hat{T} = \hat{u} \times \hat{n}$, then $\{\hat{u}, \hat{n}, \hat{T}\}$ forms a right-handed orthonormal basis ...

$$= \lim_{A\to 0} \frac{1}{A} \lim_{l\to 0} \frac{1}{2l} \cdot \int_{-l}^{l} \oint_{\Gamma\times\{h\}} \vec{F}\cdot\hat{T}\, ds\, dh$$

$$= \lim_{A\to 0} \frac{1}{A} \cdot \frac{1}{2l} \cdot 2l \oint_{\Gamma} \vec{F}\cdot d\vec{r}$$

$$= \lim_{A\to 0} \frac{1}{A} \oint_{\Gamma} \vec{F}\cdot d\vec{r}$$

This new formula for curl shows us clearly that $\operatorname{curl}\vec{F}$ tells us about how much the vector field $\vec{F}$ is spinning with respect to each direction.

Remark An interesting remark is that we can make use of this presentation of curl to conclude the Corollary: if $\vec{F}$ is a conservative vector field, then $\operatorname{curl}\vec{F}=\vec{0}$ (which we have proved earlier by other methods)

For any $\hat{u}$, we see if $\vec{F}$ is conservative,

$$\hat{u}\cdot\operatorname{curl}\vec{F} = \lim_{A\to 0}\frac{1}{A}\oint_{\Gamma}\vec{F}\cdot d\vec{r} = \lim_{A\to 0}\frac{1}{A}\cdot 0 = 0$$

So $\operatorname{curl}\vec{F}=\vec{0}$.

If we make use of the Stokes' Theorem, we can also show that if $\vec{F}$ is defined on an open simply-connected region in $\mathbb{R}^3$, then $\operatorname{curl}\vec{F}=0$ implies $\vec{F}$ is conservative.

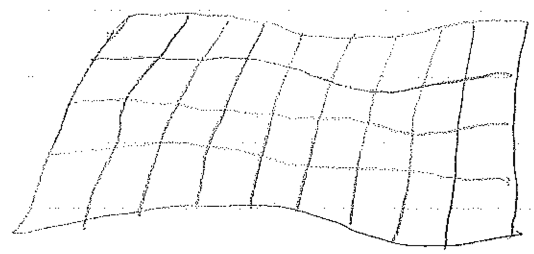
STOKES' THEOREM.

Theorem. Suppose $\vec{F}(x,y,z)$ is a vector field defined in a region containing a surface $\vec{S}$ and $\vec{F}$ has continuous partial derivatives, then:

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \oint_{\partial S} \vec{F}\cdot d\vec{r}$$

where ∂S is the boundary curve of $\vec{S}$ with counterclockwise orientation if seen from the side of $\vec{S}$ that $\hat{n}$ is pointing to, i.e. the orientation of $\vec{S}$ induces the orientation of ∂S by the right-hand rule.

Proof
$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\, dS$$



$$= \lim_{\max\Delta S\to 0} \sum_{\Delta S} \operatorname{curl}\vec{F}\cdot\hat{n}\,\Delta S$$

$$(*)\quad = \lim_{\max\Delta S\to 0} \sum_{\Delta S} \frac{1}{\Delta S}\oint_{\partial\Delta S}\vec{F}\cdot d\vec{r}\,\Delta S$$

$$= \lim_{\max\Delta S\to 0} \sum_{\Delta S} \oint_{\partial\Delta S}\vec{F}\cdot d\vec{r}$$

(Note that when the line integrals are summed up, those on the sides shared by two pieces of ΔS will be cancelled)

$$= \oint_{\partial S}\vec{F}\cdot d\vec{r}$$

Remark. A rigorous argument for $(*)$ is a standard one in analysis:
$\hat{n}\cdot\operatorname{curl}\vec{F} = \lim_{A\to 0} \frac{1}{A}\oint_{\Gamma}\vec{F}\cdot d\vec{r}$ implies there exists $\varepsilon(A) > 0$

such that $\hat{n}\cdot\operatorname{curl}\vec{F}\cdot A = \oint_{\Gamma}\vec{F}\cdot d\vec{r} + \varepsilon(A)A$ and $\varepsilon(A)\to 0$ as $A\to 0$. Then

$$\sum_{\Delta S}\operatorname{curl}\vec{F}\cdot\hat{n}\,\Delta S = \sum_{\partial\Delta S}\left(\oint_{\Gamma}\vec{F}\cdot d\vec{r} + \varepsilon(\Delta S)\Delta S\right) = \oint_{\partial S}\vec{F}\cdot d\vec{r} + \sum_{\partial\Delta S}\varepsilon(\Delta S)\Delta S$$

Taking the limit as $\max \Delta S \to 0$, we get

$$\iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, dS = \oint_{\partial S} \vec{F} \cdot d\vec{r} + \lim_{\max \Delta S \to 0} \sum \mathcal{E}(\Delta S)\Delta S$$

and $\displaystyle\lim_{\max \Delta S \to 0} \sum \mathcal{E}(\Delta S)\Delta S \leq \lim_{\max \Delta S \to 0} \left(\max_{\Delta S} \mathcal{E}(\Delta S)\right) \cdot \sum_{\Delta S} \Delta S = 0$ since

$\max \mathcal{E}(\Delta S) \to 0$ and $\sum_{\Delta S} \Delta S$ is the area of S, so bounded.

Corollary. If S is a closed surface, $\vec{F}$ is a vector field on S, then

$$\oiint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = 0$$

Proof. If S is closed, then $\partial S = \phi$. So by Stokes' Theorem:

$$\oiint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_\phi \vec{F} \cdot d\vec{r} = 0$$

Corollary. (Green's Theorem) $\vec{F}(x,y) = (P(x,y), Q(x,y))$ is a vector field on a curve C, with S the region enclosed by C. If $\vec{F}$ has continuous partial derivatives, then

$$\iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \oint_C \vec{F} \cdot d\vec{r}$$

Proof. We regard $\vec{F}$ as a vector field in $\mathbb{R}^3$: $\vec{F}(x,y,z) = (P(x,y), Q(x,y), 0)$

Then $\operatorname{Curl} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

By Stokes' Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_S \left(0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \cdot (0,0,1)\, dA$$

$$= \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

Proposition. If $\vec{F}(x,y,z)$ is a vector field defined on a simply-connected region $R \subseteq \mathbb{R}^3$, then $\operatorname{curl}\vec{F} = \vec{0}$ on R implies $\vec{F}$ is a conservative vector field

Remark. We can interpret "simply-connected" region in $\mathbb{R}^3$ to be a region R that for any closed curve $C \subseteq R$, C bounds some surface $S \subseteq R$ that has no hole

Proof. For any closed curve $C \subset R$, C bounds some surface $S \subseteq R$ that has no hole. So by Stokes' Theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = 0$$

Now we also obtain a geometric proof of the fact that $\operatorname{curl}(\nabla f) = \vec{0}$:

For any surface $\vec{S}$, $\iint_S \operatorname{curl}(\nabla f)\cdot d\vec{S} = \oint_{\partial S} \nabla f \cdot d\vec{r} = 0$

since ∇f is always a conservative vector field.

Because the above equality holds for any $\vec{S}$, we conclude the integrand $\operatorname{curl}(\nabla f) = \vec{0}$