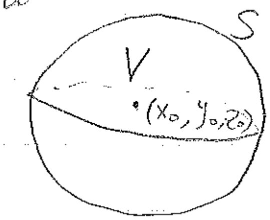
CURL

Definition. If $\vec{F}(x,y,z)$ is a vector field defined in a neighbourhood of a point (x_0, y_0, z_0), V is a region in $\mathbb{R}^3$ such that (x_0, y_0, z_0) is an interior point of V, with boundary of V to be a surface S, outward oriented. Then define the Curl of $\vec{F}$ at (x_0, y_0, z_0) to be the vector

$$\text{Curl}\,\vec{F}(x_0, y_0, z_0) = -\lim_{\text{Vol}(V)\to 0} \frac{\oiint_S \vec{F}\times d\vec{S}}{V}$$

if the limit exists. We then also get a corresponding vector field Curl$\vec{F}$ for the given vector field $\vec{F}$.



Proposition. If $\vec{F}$ and $\vec{G}$ are vector fields, $\lambda, \mu \in \mathbb{R}$, then $\text{Curl}(\lambda\vec{F} + \mu\vec{G}) = \lambda\,\text{Curl}(\vec{F}) + \mu\,\text{Curl}(\vec{G})$.

Proof. It follows directly from the definition of Curl and the fact that cross product is distributive.

Proposition. If $f(x,y,z)$ is a scalar function and $\vec{u}$ is a constant vector field, then $\text{Curl}(f\vec{u}) = \nabla f \times \vec{u}$

Proof.
$$\text{Curl}(f\vec{u}) = -\lim_{\text{Vol}(V)\to 0} \frac{\oiint_S f\vec{u}\times d\vec{S}}{V} = -\lim_{\text{Vol}(V)\to 0} \frac{\oiint_S f\vec{u}\times \hat{n}\,dS}{V}$$

$$= -\lim_{\text{Vol}(V)\to 0} \vec{u}\times \frac{\oiint_S f\hat{n}\,dS}{V}$$

$$= -\vec{u}\times\nabla f = \nabla f\times\vec{u}$$

Proposition. If $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is a vector field, then $\text{Curl}\,\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$

Proof. $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, so by the previous propositions.

$$\begin{aligned}
\text{Curl}\,\vec{F} &= \text{Curl}(P\vec{i} + Q\vec{j} + R\vec{k}) \\
&= \text{Curl}(P\vec{i}) + \text{Curl}(Q\vec{j}) + \text{Curl}(R\vec{k}) \\
&= \nabla P \times \vec{i} + \nabla Q \times \vec{j} + \nabla R \times \vec{k} \\
&= \left(\frac{\partial P}{\partial x}, \frac{\partial P}{\partial y}, \frac{\partial P}{\partial z}\right) \times (1,0,0) + \left(\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}\right) \times (0,1,0) + \\
&\quad \left(\frac{\partial R}{\partial x}, \frac{\partial R}{\partial y}, \frac{\partial R}{\partial z}\right) \times (0,0,1) \\
&= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)
\end{aligned}$$

Remark. A good way for memorizing the above result is that

$$\text{Curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{\nabla} \times \vec{F}$$

Proposition $\text{Curl}(\nabla f) = \vec{0}$ for any smooth function f.

Proof. $\text{Curl}(\nabla f) = \text{Curl}\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$$= \left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right), \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right), \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right)\right)$$

Corollary. If $\vec{F}$ is a conservative vector field, then $\text{curl}\,\vec{F} = \vec{0}$