OTHER TYPES OF INTEGRAL

Definition. Given a path C and a scalar function $\phi(x,y,z)$ defined on some region containing C, defined the line integral of ϕ along C to be

$$\int_C \phi\, d\vec{r} = \int_a^b \phi(\vec{r}(t))\, \vec{r}'(t)\, dt$$

where $\vec{r}(t) : [a,b] \to \mathbb{R}^3$ is a parameterization of C

Note the above integral should leads to a vector as a result, since $\phi(\vec{r}(t))\,\vec{r}'(t)$ is a vector.

Example. $\int_C x+y^2\, d\vec{r}$ where C is the parabola $y=x^2$ in the plane $z=0$, connecting $(0,0,0)$ and $(1,1,0)$

C can be parameterized by $\vec{r}(t) = (t, t^2, 0)$, $t \in [0,1]$.

$$\begin{aligned}
\int_C x+y^2\, d\vec{r} &= \int_0^1 (t+(t^2)^2)\, \vec{r}'(t)\, dt \\
&= \int_0^1 (t+t^4)\cdot(1, 2t, 0)\, dt \\
&= \int_0^1 (t+t^4,\ 2t^2+2t^5, 0)\, dt \\
&= \left(\int_0^1 t+t^4\, dt,\ \int_0^1 2t^2+2t^5\, dt,\ \int_0^1 0\, dt\right) \\
&= \left(\tfrac{7}{10}, 1, 0\right)
\end{aligned}$$

Definition. Given a path C parameterized by $\vec{r}(t)$, $t \in [a,b]$, and a vector field $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ define the cross integral of $\vec{F}$ along C to be

$$\int_C \vec{F} \times d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \times \vec{r}\,'(t)\, dt$$

Similar as the previous case, the result of this integral is also a vector.

We can also define the corresponding concepts on oriented surfaces, and an important one is the following:

Definition. Given an oriented surface $\vec{S}$ with orientation $\hat{n}$, if $\vec{S}$ can be parameterized by $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$ with domain $(s,t) \in D$, and $\vec{F}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ is a vector field on $\vec{S}$, we define the cross integral of $\vec{F}$ along $\vec{S}$ to be:

$$\iint_S \vec{F} \times d\vec{S} = \iint_S \vec{F}(\vec{r}(s,t)) \times \hat{n}\, dS$$

$$= \iint_D \vec{F}(\vec{r}(s,t)) \times \hat{n}\, \left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right| dA$$

When $\hat{n}$ agrees with the orientation induced by parameterization, i.e. $\hat{n} = \dfrac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right|}$, we can further write

$$\iint_S \vec{F} \times d\vec{S} = \iint_D \vec{F}(\vec{r}(s,t)) \times \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) dA$$

Example. Let $\vec{F}(x,y,z) = (x,y,z)$, and $\vec{S}$ is the unit sphere $x^2+y^2+z^2=1$ with outward orientation.

$$\iint_S \vec{F} \times d\vec{S} = \iint_D \vec{F}(\vec{r}(\varphi,\theta)) \times \left(\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}\right) dA$$

$$= \iint_D (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi) \times (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)\, dA$$

$$= \iint_D (0,0,0)\, dA$$

$$= (0,0,0)$$

Example. Let $\vec{F}(x,y,z) = (-y, x, 0)$, and $\vec{S}$ is the unit sphere $x^2+y^2+z^2=1$ with outward orientation.

$$\iint_S \vec{F} \times d\vec{S} = \iint_D \vec{F}(\vec{r}(\varphi,\theta)) \times \left(\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}\right) dA$$

$$= \iint_D (-\sin\varphi\sin\theta, \sin\varphi\cos\theta, 0) \times (\sin^2\varphi\cos\theta, \sin^2\varphi\sin\theta, \sin\varphi\cos\varphi)\, dA$$

$$= \iint_D (\sin^2\varphi\cos\varphi\cos\theta, \sin^2\varphi\cos\varphi\sin\theta, -\sin^3\varphi)\, dA$$

$$= \left(\int_0^{2\pi}\int_0^{\pi} \sin^2\varphi\cos\varphi\cos\theta\, d\varphi\, d\theta, \int_0^{2\pi}\int_0^{\pi} \sin^2\varphi\cos\varphi\sin\theta\, d\varphi\, d\theta, \int_0^{2\pi}\int_0^{\pi} -\sin^3\varphi\, d\varphi\, d\theta\right)$$

$$= \left(0, 0, -\frac{8}{3}\right)$$

We can also integrate a scalar function on an oriented surface:

Definition If $f(x,y,z)$ is a scalar function defined on an oriented surface S, define:

$$\iint_S f\, d\vec{S} = \iint_S f\hat{n}\, dS = \iint_D f(\vec{r}(s,t))\, \hat{n}(\vec{r}(s,t)) \left|\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right| dA$$

if $\vec{S}$ is parameterized by $\vec{r}(s,t)$ with domain D.

If $\vec{r}(s,t)$ agrees with the orientation of $\vec{S}$, we see

$$\iint_S f\, d\vec{S} = \iint_D f(\vec{r}(s,t)) \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right) dA$$

Example. If $\vec{S}$ is the unit sphere $x^2+y^2+z^2=1$ with outward orientation, and $f(x,y,z) = z$, then

$$\iint_S f\, d\vec{S} = \iint_D \cos\varphi \left(\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}\right) dA$$

$$= \iint_D (\sin^2\varphi \cos\varphi \cos\theta,\ \sin^2\varphi \cos\varphi \sin\theta,\ \sin\varphi \cos^2\varphi)\, dA$$

$$= \left(\iint_D \sin^2\varphi \cos\varphi \cos\theta\, dA,\ \iint_D \sin^2\varphi \cos\varphi \sin\theta\, dA,\ \iint_D \sin\varphi \cos^2\varphi\, dA\right)$$

$$= \left(0, 0, \frac{4\pi}{3}\right)$$

We can reinterpret the concept of gradient by considering integral of this form:

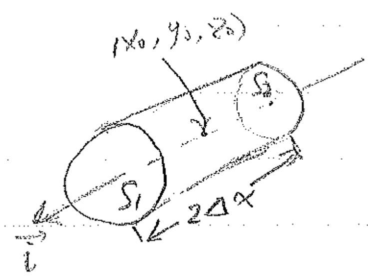
Definition $f(x,y,z)$ is a scalar function, V is some region in $\mathbb{R}^3$ with (x_0, y_0, z_0) in its interior, and $\vec{S}$ is the boundary of V with outward orientation. Then define the gradient of f at (x_0, y_0, z_0) to be the vector

$$\nabla f = \lim_{V \to 0} \frac{\oiint_S f \, d\vec{S}}{V}$$

if the limit exists. So we get a vector field ∇f, called the gradient of f.

Proposition $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

Proof We're going to show that $\nabla f \cdot \vec{i} = \frac{\partial f}{\partial x}$ at each point.
Let V be a cylindrical region with core in the direction of $\vec{i}$.



$$\nabla f(x_0, y_0, z_0) \cdot \vec{i} = \left(\lim_{V \to 0} \frac{\oiint_S f \, d\vec{S}}{V}\right) \cdot \vec{i}$$

$$= \lim_{V \to 0} \frac{\oiint_S f \, \hat{n} \, dS}{V} \cdot \vec{i}$$

$$= \lim_{V \to 0} \frac{\oiint_S f \, \vec{i} \cdot \hat{n} \, dS}{V}$$

Note for the curved parts of the cylinder, $\hat{n} \perp \vec{i}$. on the two disks, S_1 & S_2, $\hat{n} \parallel \vec{i}$

$$= \lim_{V \to 0} \frac{\iint_{S_1} f \, dS - \iint_{S_2} f \, dS}{V}$$

$$= \lim_{V \to 0} \frac{\iint_{D_r} f(x_0 + \Delta x, y, z) - f(x_0 - \Delta x, y, z) \, dA}{\text{Area}(D_r) \cdot 2\Delta x}$$

where D_r is the disk on y-z plane centered at (y_0, z_0) with radius r.

$$= \lim_{V \to 0} \frac{\iint_{D_r} \frac{\partial f}{\partial x}(x_0, y, z) \cdot 2\Delta x + \varepsilon(y,z) \Delta x \, dA}{\iint_{D_r} 1 \, dA \cdot 2\Delta x}$$

$$= \lim_{r \to 0} \left(\frac{\iint_{D_r} \frac{\partial f}{\partial x}(x_0, y, z) \, dA}{\iint_{D_r} 1 \, dA} + \lim_{\Delta x \to 0} \frac{\iint_{D_r} \varepsilon(y,z) \, dA}{2 \iint_{D_r} 1 \, dA} \right)$$

$$= \frac{\partial f}{\partial x}(x_0, y_0, z_0)$$