Definition. If $\vec{F}(x,y,z)$ is a vector field defined on a region containing an oriented surface $\vec{S}$ (a surface S with continuous choice of unit normal vectors $\vec{n}$), define the flux of $\vec{F}$ through $\vec{S}$ to be

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS$$

when S is parameterized by $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$, we know $\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}$ is a normal vector of the surface, so a choice of unit normal vector is

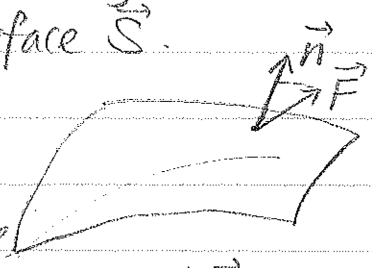
$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left|\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right|}$$

We then can evaluate the integral based on this orientation:

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(s,t))\cdot \frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left|\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right|} \left|\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right| dA$$

$$= \iint_D \vec{F}(\vec{r}(s,t))\cdot\left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right) dA$$

The practical meaning of the flux is to study the amount of some flow $\vec{F}$ through some surface $\vec{S}$. The observation is only the component of $\vec{F}$ that is parallel to $\vec{n}$ (i.e normal to the surface) passes through the surface



and if $|\vec{n}|=1$, this component is computed by $\vec{F}\cdot\vec{n} = |\vec{F}|\cos\theta$

Then the total flux is to sum up the flux at each point along the surface, via the language of integral.

This concept of flux plays an important role in several branches of physics, such as fluid dynamics, heat transfer, and electromagnetic dynamics.

Notation. When the surface is closed (i.e. bounds some volume), and $\vec{n}$ is taken to be the outgoing unit normal vector, we write the flux as

$$\oiint_S \vec{F} \cdot d\vec{S}$$

Example. Find the flux of $\vec{F}(x,y,z) = (z, y, x)$ across the unit sphere $x^2+y^2+z^2=1$: $\oiint_S \vec{F} \cdot d\vec{S}$

We can parameterize the sphere by

$\vec{r}(\varphi,\theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$, $0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$

Then

$$\frac{\partial\vec{r}}{\partial\varphi} \times \frac{\partial\vec{r}}{\partial\theta} = (\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi)$$

So
$$\frac{\frac{\partial\vec{r}}{\partial\varphi} \times \frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\varphi} \times \frac{\partial\vec{r}}{\partial\varphi}\right|} = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

We see at $(1,0,0)$, $\varphi = \frac{\pi}{2}$, $\theta = 0$. So

$$\frac{\frac{\partial\vec{r}}{\partial\varphi} \times \frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\varphi} \times \frac{\partial\vec{r}}{\partial\varphi}\right|} = (1, 0, 0),$$ which is outward going.

So $\dfrac{\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}}{\left|\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}\right|}$ gives the orientation that we want.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\varphi,\theta)) \cdot \left(\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}\right) dA$$

$$= \int_0^{2\pi}\int_0^{\pi} (\cos\varphi, \sin\varphi\sin\theta, \sin\varphi\cos\theta) \cdot (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} 2\sin^2\varphi\cos\varphi\cos\theta + \sin^3\varphi\sin^2\theta \, d\varphi\, d\theta$$

$$= 2\int_0^{\pi} \sin^2\varphi\cos\varphi\, d\varphi \int_0^{2\pi}\cos\theta\, d\theta + \int_0^{\pi}\sin^3\varphi\, d\varphi \int_0^{2\pi} \sin^2\theta\, d\theta$$

$$= \frac{4\pi}{3}$$

Example. When $\vec{S}$ is the graph of the function $z = g(x,y)$ defined on $D \subseteq \mathbb{R}^2$ with upward orientation, observe that the parameterization $\vec{r}(x,y) = (x, y, g(x,y))$ gives the orientation $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right)$ which is indeed upward, so for a vector field $\vec{F} = (P, Q, R)$.

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(x,y)) \cdot \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right) dA$$

$$= \iint_D \left(P(\vec{r}(x,y)), Q(\vec{r}(x,y)), R(\vec{r}(x,y))\right) \cdot \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) dA$$

$$= \iint_D -\frac{\partial g}{\partial x} P(x,y,g(x,y)) - \frac{\partial g}{\partial y} Q(x,y,g(x,y)) + R(x,y,g(x,y))\, dA$$