SURFACE INTEGRAL

In order to define surface integrals, we need to first find a way to describe a surface:

Definition. A parametric surface S is a function $\mathbb{R}^2 \supseteq D \longrightarrow \mathbb{R}^3$ $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$. We usually require $\vec{r}$ to be injective when restricted to the interior of D.

Example. If S is the surface of the graph of $z = f(x,y)$, we can write it as $\vec{r}(s,t) = (s, t, f(s,t))$

Example. The cylinder $x^2+y^2=4$, $0 \le z \le 1$ can be parameterized by $\vec{r}(\theta, z) = (2\cos\theta, 2\sin\theta, v)$, $0 \le \theta \le 2\pi$, $0 \le v \le 1$

Example. The unit sphere $x^2+y^2+z^2=1$ can be parameterized by $\vec{r}(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$, $0 \le \varphi \le \pi$, $0 \le \theta \le 2\pi$

When we study curves, we need to make use of tangent lines; now we need to look into the tangent planes in order to study surfaces.

Given a surface $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$, we will find its tangent plane by finding two tangent vectors which will span the plane.

For a fixed (s_0, t_0), consider the curves:

$\vec{\alpha}(s) = \vec{r}(x(s,t_0), y(s,t_0), z(s,t_0))$

$\vec{\beta}(t) = \vec{r}(x(s_0,t), y(s_0,t), z(s_0,t))$

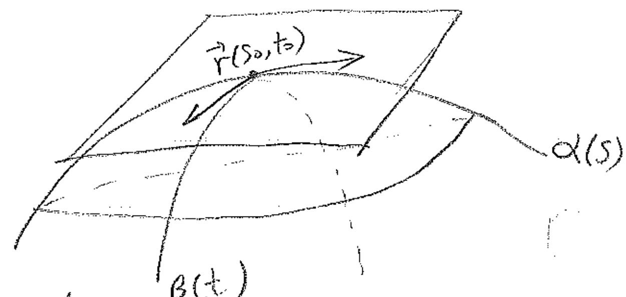
$\vec{\alpha}(s) \subseteq \vec{r}(s,t)$, $\vec{\beta}(t) \subseteq \vec{r}(s,t)$.

$$\vec{\alpha}'(s_0) = \frac{\partial \vec{r}}{\partial s}(s_0,t_0) = \left(\frac{\partial x}{\partial s}(s_0,t_0), \frac{\partial y}{\partial s}(s_0,t_0), \frac{\partial z}{\partial s}(s_0,t_0)\right)$$

$$\vec{\beta}'(t_0) = \frac{\partial \vec{r}}{\partial t}(s_0,t_0) = \left(\frac{\partial x}{\partial t}(s_0,t_0), \frac{\partial y}{\partial t}(s_0,t_0), \frac{\partial z}{\partial t}(s_0,t_0)\right)$$

both $\vec{\alpha}'(s_0)$ and $\vec{\beta}'(t_0)$ are in the tangent plane of $\vec{r}(s,t)$ at $\vec{r}(s_0,t_0)$, so if they're not parallel, $\vec{\alpha}'(s_0) \times \vec{\beta}'(t_0)$ gives a normal vector of the tangent plane at $\vec{r}(s_0,t_0)$.

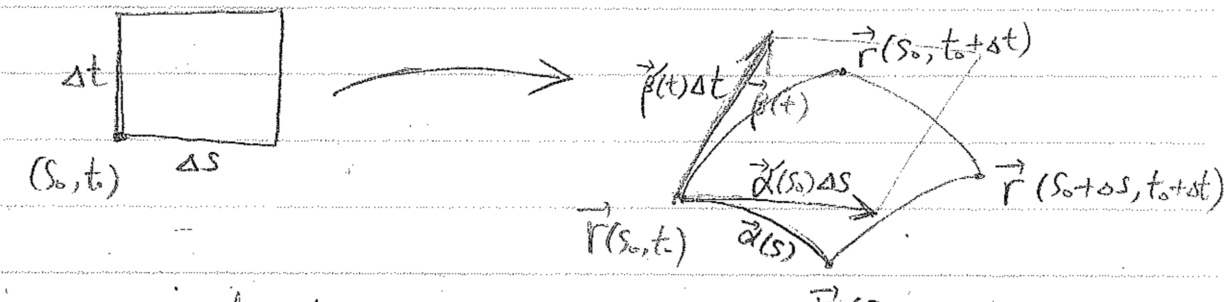
Definition. If S is parameterized by $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$ with domain $D \subseteq \mathbb{R}^2$. the area of the surface is defined by the double integral



$$A(S) = \iint_D \left| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right| dA$$

We will see this is a reasonable definition: One way to try to define the area of a given surface is to first approximate the surface by many small pieces of planes, then the limit of the sum of the area of those small pieces as the size of pieces approaching 0 should be a good description of the area of the surface.

So we first take a small square in the st-plane with lower left corner at (s_0, t_0), then try to approximate the area of its image under $\vec{r}(s,t)$ by the tangent plane passing through $\vec{r}(s_0,t_0)$.



when Δs and Δt are small, recall that the length of $\vec{\alpha}(s)$ for $s\in[s_0, s_0+\Delta s]$ can be estimated by $|\vec{\alpha}'(s_0)|\cdot\Delta s$, and similarly the length of $\vec{\beta}(t)$ for $t\in[t_0, t_0+\Delta t]$ can be estimated by $|\vec{\beta}'(t_0)|\Delta t$. and the part of the surface is close to the tangent plane, so we can estimate the area of that piece of the surface by the area of the parallelogram bounded by $\vec{\alpha}'(s_0)\Delta s$ and $\vec{\beta}'(t)\Delta t$, which is

$$|\vec{\alpha}'(s_0)\Delta s \times \vec{\beta}'(t_0)\Delta t| = |\vec{\alpha}'(s_0)\times\vec{\beta}'(t_0)|\,\Delta s\Delta t$$
$$= \left|\frac{\partial\vec{r}}{\partial s}(s_0,t_0)\times\frac{\partial\vec{r}}{\partial t}(s_0,t_0)\right|\Delta s\,\Delta t$$

Finally we sum up all these small pieces by integration, to get
$$\iint_D \left|\frac{\partial\vec{r}}{\partial s}\times\frac{\partial\vec{r}}{\partial t}\right| dA$$

Example. The surface area of the sphere $x^2+y^2+z^2=1$ is computed as follows:
We parameterize the sphere by
$$\vec{r}(\varphi,\theta) = (\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)\ ;\ \varphi\in[0,\pi],\ \theta\in[0,2\pi]$$

$$\frac{\partial\vec{r}}{\partial\varphi} = (\cos\varphi\cos\theta,\ \cos\varphi\sin\theta,\ -\sin\varphi)$$

$$\frac{\partial \vec{r}}{\partial \theta} = (-\sin\varphi\sin\theta,\ \sin\varphi\cos\theta,\ 0)$$

$$\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta} = (\sin^2\varphi\cos\theta,\ \sin^2\varphi\sin\theta,\ \sin\varphi\cos\varphi)$$

$$\left|\frac{\partial \vec{r}}{\partial \varphi} \times \frac{\partial \vec{r}}{\partial \theta}\right| = \sin\varphi$$

So the surface area is

$$\int_0^{2\pi}\int_0^{\pi} \sin\varphi\, d\varphi\, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi} \sin\varphi\, d\varphi = 2\pi \cdot 2 = 4\pi$$

Now based on the definition of surface area, we can define different kinds of surface integrals.

Definition. If $f(x,y,z)$ is a function defined on a surface S parameterized by $\vec{r}(s,t) = (x(s,t), y(s,t), z(s,t))$ on $D \subseteq \mathbb{R}^2$, define the surface integral of f on S to be:

$$\iint_S f(x,y,z)\, dS = \iint_D f(\vec{r}(s,t)) \left|\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right| dA$$

Remark. From the Riemann Sum approach, the surface integral should be defined by the following process:

$$\iint_S f(x,y,z)\, dS = \lim_{\max \Delta S \to 0} \sum_{i \in I} f(x_i^*, y_i^*, z_i^*) \cdot \Delta S_i$$

$$= \lim_{\max \Delta A \to 0} \sum_{i \in I} f(\vec{r}(s_i^*, t_i^*)) \left|\frac{\partial \vec{r}}{\partial s}(s_i^*, t_i^*) \times \frac{\partial \vec{r}}{\partial t}(s_i^*, t_i^*)\right| \Delta A$$

$$= \iint_D f(\vec{r}(s,t)) \left|\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}\right| dA$$

Example. A thin sphere $x^2+y^2+z^2=1$ made of some mixed medals has density function $\rho(x,y,z)=x^2$. Compute its mass.

$$\vec{r}(\varphi,\theta)=(\sin\varphi\cos\theta,\ \sin\varphi\sin\theta,\ \cos\varphi)$$

$\left|\frac{\partial\vec{r}}{\partial\varphi}\times\frac{\partial\vec{r}}{\partial\theta}\right|=\sin\varphi$ as we computed in the previous example.

$$\text{Mass}=\iint_S \rho(x,y,z)\,dS=\int_0^{2\pi}\int_0^{\pi}(\sin\varphi\cos\theta)^2\sin\varphi\,d\varphi\,d\theta$$

$$=\int_0^{2\pi}\cos^2\theta\,d\theta\cdot\int_0^{\pi}\sin^3\varphi\,d\varphi$$

$$=\frac{4}{3}\pi$$

Example. If S is the graph of the function $z=g(x,y)$ on $D\subseteq\mathbb{R}^2$. We can parameterize S by $\vec{r}(x,y)=(x,y,g(x,y))$

$$\frac{\partial\vec{r}}{\partial x}=\left(1,0,\frac{\partial g}{\partial x}\right),\quad \frac{\partial\vec{r}}{\partial y}=\left(0,1,\frac{\partial g}{\partial y}\right)$$

$$\frac{\partial\vec{r}}{\partial x}\times\frac{\partial\vec{r}}{\partial y}=\left(-\frac{\partial g}{\partial x},-\frac{\partial g}{\partial y},1\right).$$

$$\left|\frac{\partial\vec{r}}{\partial x}\times\frac{\partial\vec{r}}{\partial y}\right|=\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}$$

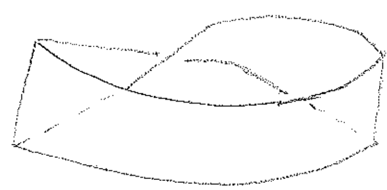
So $\displaystyle\iint_S f(x,y,z)\,dS=\iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2+1}\,dA$

Next we are going to define another kind of surface integral, which is to study the amound of flow passing through a surface.

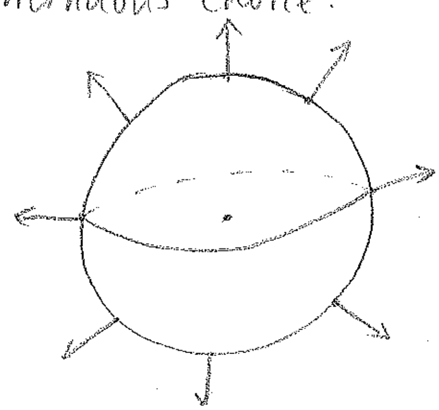
Definition. An oriented surface $\vec{S}$ is a surface with a continuous choice of unit normal vectors. $\vec{n}$

Definition. If it's possible to have a continuous choice of unit normal vectors for a surface S, we say S is orientable.

Example. The Mobius Stripe is NOT orientable:



Example. The sphere is orientable, we can choose the outgoing unit normal vector at every point, which is a continuous choice:



Remark. A convension is that when we regard a closed surface as an oriented surface, if not specified, we choose the outgoing unit normal vector as the orientation.