IMPLICIT FUNCTION THEOREM

Theorem. (Implicit Function Theorem)

$F: \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is continuously differentiable in an open set containing $(a,b) \in \mathbb{R}^n \times \mathbb{R}^m$, and $F(a,b) = 0$.

Let $M = \left(\frac{\partial f^i}{\partial x_{n+j}}(a,b)\right)$ be the $m \times m$ matrix.

If $\det M \neq 0$, then there exists open set $a \in A \subseteq \mathbb{R}^n$ and open set $b \in B \subseteq \mathbb{R}^m$ such that there is a unique function

$$G: A \longrightarrow B$$

such that $F(x, G(x)) = 0$, and G is differentiable.

Proof. Define

$$\begin{aligned} \Phi: \mathbb{R}^n \times \mathbb{R}^m &\longrightarrow \mathbb{R}^n \times \mathbb{R}^m \\ (x, y) &\longmapsto (x, F(x,y)) \end{aligned}$$

$$D\Phi(a,b) = \left(\begin{array}{c|c} I_n & O \\ \hline * & M \end{array}\right)$$

$$\det D\Phi(a,b) = \det I_n \cdot \det M \neq 0$$

The Inverse Function Theorem implies there is an open set $(a,b) \in A \times B \subseteq \mathbb{R}^n \times \mathbb{R}^m$ and open set $\Phi(a,b) = (a, F(a,b)) = (a,0) \in W \subseteq \mathbb{R}^n \times \mathbb{R}^m$ such that $\Phi: A \times B \longrightarrow W$ has differentiable inverse $\Psi: W \longrightarrow A \times B$.

We can write $\Psi(x,y) = (x, K(x,y))$ for some differentiable function $K : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$

Define $\pi : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$

$$(x, y) \longmapsto y$$

The $\pi \circ \Phi(x,y) = \pi((x, F(x,y))) = F(x,y)$

So:

$$F(x, K(x,y)) = F(\Psi(x,y)) = \pi \circ \Phi \circ \Psi(x,y) = \pi(x,y) = y$$

This implies $F(x, K(x,0)) = 0$

Let $G(x) = K(x,0)$, we get $F(x, G(x)) = 0$

Corollary. The derivative of the implicit function G can be computed by implicit differentiation:

$$0 = \frac{\partial f_i}{\partial x_j}(x, g(x)) + \sum_{\alpha=1}^{m} \frac{\partial f_i}{\partial x_{n+\alpha}}(x, g(x)) \cdot \frac{\partial g_\alpha}{\partial x_j}(x)$$

In matrix Form, if

$$DF(x, g(x)) = \left.\left[\underbrace{DF_x}_{n} \;\vdots\; \underbrace{DF_y}_{m} \right]\right\} m$$

Then $DF_x(x, G(x)) + DF_y(x, G(x)) \cdot DG(x) = 0$

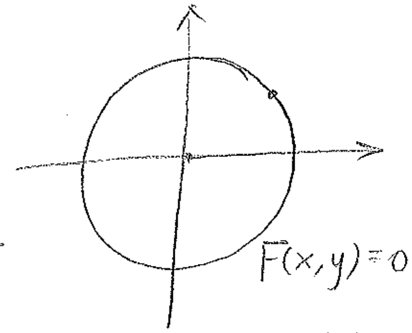
$$DG(x) = -DF_y(x, G(x))^{-1} \cdot DF_x(x, G(x))$$

Example. $F: \mathbb{R}^2 \to \mathbb{R}$ is $F(x,y) = x^2 + y^2 - 1$

$$DF(x,y) = \left[\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right] = [2x, 2y]$$

$M = [2y] \neq 0$ for any (a,b) such that $a^2 + b^2 - 1 = 0$ and $b \neq 0$

so y is a function of x near (a,b) on this level set $F(x,y) = 0$, i.e. $y = G(x)$ such that $F(x, G(x)) = 0$



The derivative is computed by

$$DG(x) = -[2y]^{-1} \cdot [2x] = -\frac{x}{y} = -\frac{x}{G(x)}$$

Note that this agrees with what we have learned about implicit differentiation in Calculus.

Lemma. $F: \mathbb{R}^2 \to \mathbb{R}$ is a smooth function. If $\nabla F(x_0, y_0) \neq \vec{0}$ for some $(x_0, y_0) \in \mathbb{R}^2$, then the level set $F(x,y) = F(x_0, y_0)$ locally is a differentiable curve near (x_0, y_0)

Proof. If $\nabla F(x_0, y_0) = \left(\frac{\partial F}{\partial x}(x_0, y_0), \frac{\partial F}{\partial y}(x_0, y_0)\right) \neq \vec{0}$, then at least one of the coordinates is nonzero. Without loss of generality we may assume $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$.

Then by the implicit function theorem, there is a function $g: \mathbb{R} \to \mathbb{R}$ such that $F(x, g(x)) = F(x_0, y_0)$ for x near x_0

This implies the level set $F(x,y) = F(x_0, y_0)$ near (x_0, y_0) is the graph of the function $y = g(x)$, hence it's the curve

$\vec{r}(t) = (t, g(t))$. near $t = x_0$.

Proposition. $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a smooth function and $F(x,y) = C$ is a level set. If $\nabla F \neq \vec{0}$ on $F(x,y) = C$ everywhere, then $F(x,y) = C$ represents a differentiable curve in $\mathbb{R}^2$

Proof. By the Lemma, we see locally it's a curve everywhere so it's a curve itself.

Similar arguments lead to the following results:

Proposition. $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a smooth function, and $F(x,y,z) = C$ is a level set. If $\nabla F(x,y,z) \neq \vec{0}$ on $F(x,y,z) = C$ everywhere, then $F(x,y,z) = C$ represents a differentiable surface in $\mathbb{R}^2$

Example. In $\mathbb{R}^2$, $F(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2}$, $(ab \neq 0)$. Then

For any $C > 0$: If $F(x,y) = C$, then $(x,y) \neq (0,0)$. So $\nabla F = \left(\frac{2x}{a^2}, \frac{2y}{b^2}\right) \neq 0$ on $F(x,y) = C$. we get $F(x,y) = C$ is a differentiable curve.

Indeed, $F(x,y) = C$ represents an ellipse in this case.