INVERSE FUNCTION THEOREM

Definition. $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $F(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ is called continuously differentiable at $a \in \mathbb{R}^n$ if all $\frac{\partial f_i}{\partial x_j}$ exist in a neighbourhood of a and continuous at a.

Lemma. $A \subseteq \mathbb{R}^n$ is a rectangle. $F: A \to \mathbb{R}^n$ is continuously differentiable on A. If there is $M > 0$ such that $\forall\, x \in A \setminus \partial A$ $\left|\frac{\partial f_i}{\partial x_j}(x)\right| \leq M$, then:

$$|F(x) - F(y)| \leq n^2 M |x - y| \quad \text{for all } x, y \in A$$

Proof. $\forall\, x, y \in A$.

$$f_i(y) - f_i(x) = \sum_{j=1}^{n} \left[f_i(y_1, \ldots, y_j, x_{j+1}, \ldots, x_n) - f_i(y_1, \ldots, y_{j-1}, x_j, \ldots, x_n) \right]$$

$$= \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(y_1, \ldots, y_{j-1}, z_j, x_{j+1}, \ldots, x_n) \cdot (y_j - x_j)$$

for some z_j between x_j & y_j, by Mean Value Theorem.

$$|F(x) - F(y)| \leq \sum_{i=1}^{n} |f_i(y) - f_i(x)|$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \frac{\partial f_i}{\partial x_j}(y_1, \ldots, y_{j-1}, z_j, x_{j+1}, \ldots, x_n) \right| \cdot |y_j - x_j|$$

$$\leq \sum_{j=1}^{n} nM \cdot |y_j - x_j| \leq nM \cdot n\, |y - x| = n^2 M |y - x|$$

Definition If $F:\mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $a \in \mathbb{R}^n$, define the Jacobian of F at a to be the $m\times n$ matrix

$$DF(a) = \left(\frac{\partial f_i}{\partial x_j}(a)\right)_{m\times n}$$

Theorem. (Inverse Function Theorem)

If $F:\mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in an open set containing $a\in\mathbb{R}^n$, and $\det DF(a) \neq 0$, then there exists an open set V containing a and an open set W containing $f(a)$ such that $F: V \to W$ has inverse $F^{-1}: W \to V$ which is also differentiable and $\forall\, y \in W$

$$DF^{-1}(y) = \left[DF(F^{-1}(y))\right]^{-1}$$

Proof. Let L denote the linear transformation represented by $DF(a) = \left(\frac{\partial f_i}{\partial x_j}\right)$, $L:\mathbb{R}^n \to \mathbb{R}^n$ is invertible since $\det DF(a) \neq 0$

Then $D(L^{-1}\circ F)(a) = D(L^{-1})\, DF(a)$ (by the Chain Rule)

$$\begin{aligned} &= L^{-1}\cdot DF(a) \\ &= DF(a)^{-1}\cdot DF(a) \\ &= I_n \end{aligned}$$

If we can prove the Theorem is true for $L^{-1}\circ F$, then since L is invertible, we'll get the Theorem is true for F

The above observation implies we can assume $DF(a) = I_n$ to prove the Theorem.

F is continuously differentiable at $a \in \mathbb{R}^n$ implies

$$\lim_{x \to a} \frac{|F(x) - F(a) - DF(a)(x-a)|}{|x-a|} = 0$$

i.e. $$\lim_{x \to a} \frac{|F(x) - F(a) - (x-a)|}{|x-a|} = 0$$

So for $\varepsilon = \frac{1}{2}$, $\exists\ \delta > 0$ such that
$$0 < |x-a| < \delta \Rightarrow \frac{|F(x) - F(a) - (x-a)|}{|x-a|} < \frac{1}{2}$$

Suppose $0 < |b-a| < \delta$ and $F(b) = F(a)$, then

$$\frac{|\cancel{F(b) - F(a)} - (b-a)|}{|b-a|} = \frac{|b-a|}{|b-a|} = 1 > \frac{1}{2}.$$ Contradiction

So for any $0 < |x-a| < \delta$, $F(x) \neq F(a)$

We can take even smaller δ', to get $0 < |x-a| < \delta'$
Such that:
(1) $0 < |x-a| < \delta' \Rightarrow F(x) \neq F(a)$
(2) $\det DF(x) \neq 0 \quad \forall\ |x-a| < \delta'$ (since $\det DF$ is continuous)
(3) $\left|\frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(a)\right| < \frac{1}{2n^2}$ on $|x-a| < \delta'$ (since F is continuously differentiable at a)

Take U to be a rectangle inside $|x-a| < \delta'$ and $a \in U$.
Let $G(x) = F(x) - x$, and apply the Lemma, we get
for any $x, x' \in U$.

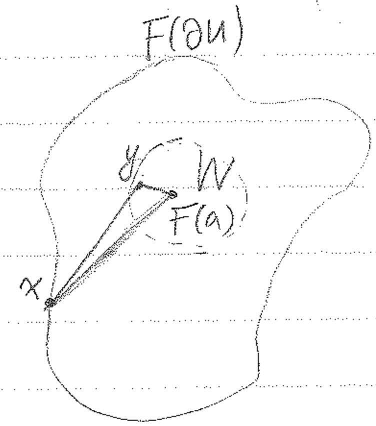
$$|G(x) - G(x')| \leq n^2 \cdot \frac{1}{2n^2} |x - x'| = \frac{1}{2} |x - x'|$$

So $|x - x'| - |F(x) - F(x')| \leq |G(x) - G(x')| \leq \frac{1}{2} |x - x'|$

$$\Rightarrow \quad |x - x'| \leq 2 |F(x) - F(x')| \text{ on } U \quad \cdots \quad (4)$$

Since $F(x)\neq F(a)\ \forall\, x\in U-\{a\}$, we see $F(\partial U)\not\ni F(a)$

Let $d=\min\limits_{x\in\partial U}|F(x)-F(a)|$.



Define $W=\{y\in\mathbb{R}^n \mid |y-F(a)|<\frac{d}{2}\}$

For any $y\in W$, any $x\in\partial U$, we get

$$\begin{aligned}|y-F(x)| &= |y-F(a)+F(a)-F(x)|\\ &\geq |F(a)-F(x)|-|y-F(a)|\\ &> d-\frac{d}{2}\\ &= \frac{d}{2}\\ &> |y-F(a)|\end{aligned}$$

i.e. $|y-F(a)|<|y-F(x)|\quad \forall\, y\in W,\ \forall\, x\in\partial U$. ----(5)

Now we are going to show for any $y\in W$, $\exists!\, x\in U$ such that $F(x)=y$:

Existence:

Let $g: U\to\mathbb{R}$ be $g(x)=|y-F(x)|^2=\sum\limits_{i=1}^{n}(y_i-f_i(x))^2$

U is a rectangle, hence compact. the continuous function $g(x)$ has a minimum x_0 on U

If $x\in\partial U$, by (5), $g(a)=|y-F(a)|^2<|y-F(x)|^2$, so $x_*\notin\partial U$, x_* is in the interior of U

We know if an interior point is a minimum, then it's a critical point. so

$$\frac{\partial g}{\partial x_j}(x_*) = 0 \ \forall \ j = 1, \dots, n$$

$$\text{i.e. } 2\sum_{i=1}^{n} (y_i - f_i(x_*)) \frac{\partial f_i}{\partial x_j}(x_*) = 0 \ \forall \ j = 1, \dots, n.$$

The matrix form is

$$2 \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x_*) & \cdots & \frac{\partial f_1}{\partial x_n}(x_*) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x_*) & \cdots & \frac{\partial f_n}{\partial x_n}(x_*) \end{pmatrix}}_{DF(x_0)} \begin{pmatrix} y_1 - f_1(x_*) \\ \vdots \\ y_n - f_n(x_*) \end{pmatrix} = 0$$

Since $\det DF(x_*) \neq 0$ by (2), we get $y_i = f_i(x_*) \ \forall \ i = 1, \dots, n$.

Uniqueness: follows directly from (4).

So we have proved $\forall \ y \in W$, $\exists ! \ x \in U \setminus \partial U$ such that $y = F(x)$.

Let $V = U \cap F^{-1}(W)$, we just showed $F: V \to W$ is a bijection, so it has inverse $F^{-1}: W \to V$.

(4) can be rewritten as $|F^{-1}(y) - F^{-1}(y')| \leq 2|y - y'| \ \forall \ y, y' \in W$. So we see F^{-1} is continuous on W.

We need to show at last F^{-1} is differentiable on W:

For any $x_0 \in V$, since F is differentiable at x_0,

we can write $F(x) = F(x_0) + DF(x_0)(x - x_0) + \varphi(x - x_0)$

such that $\lim\limits_{x \to x_0} \frac{|\varphi(x - x_0)|}{|x - x_0|} = 0$

$$[DF(x_0)]^{-1} \cdot (F(x) - F(x_0)) = [DF(x_0)]^{-1} [DF(x_0)(x - x_0) + \varphi(x - x_0)]$$

$$[DF(x_0)]^{-1} (F(x) - F(x_0)) = x - x_0 + [DF(x_0)]^{-1} \varphi(x - x_0)$$

Now for any $y_0 \in W$, if we denote $x_0 = F^{-1}(y_0)$, then $\forall y \in W$:

$$[DF(x_0)]^{-1} (F(F^{-1}(y)) - F(F^{-1}(y_0))) = F^{-1}(y) - F^{-1}(y_0) + [DF(x_0)]^{-1} \varphi(F^{-1}(y) - F^{-1}(y_0))$$

$$\Rightarrow [DF(x_0)]^{-1} (y - y_0) = F^{-1}(y) - F^{-1}(y_0) + [DF(x_0)]^{-1} \varphi(F^{-1}(y) - F^{-1}(y_0))$$

$$\Rightarrow F^{-1}(y) = F^{-1}(y_0) + [DF(x_0)]^{-1}(y - y_0) - [DF(x_0)]^{-1} \varphi(F^{-1}(y) - F^{-1}(y_0))$$

Note that $\lim\limits_{y \to y_0} \frac{|[DF(x_0)]^{-1} \varphi(F^{-1}(y) - F^{-1}(y_0))|}{|y - y_0|}$

$$= \lim_{y \to y_0} [DF(x_0)]^{-1} \cdot \frac{|\varphi(F^{-1}(y) - F^{-1}(y_0))|}{|F^{-1}(y) - F^{-1}(y_0)|} \cdot \frac{|F^{-1}(y) - F^{-1}(y_0)|}{|y - y_0|}$$

$$\leq 2 [DF(x_0)]^{-1} \lim_{y \to y_0} \frac{|\varphi(F^{-1}(y) - F^{-1}(y_0))|}{|F^{-1}(y) - F^{-1}(y_0)|}$$

$$= 0$$

since F^{-1} is continuous and $\lim\limits_{x \to x_0} \frac{|\varphi(x - x_0)|}{|x - x_0|} = 0$.

We conclude F^{-1} is differentiable at $y_0 \in W$, and

$DF^{-1}(y_0) = [DF(F^{-1}(y_0))]^{-1}$, i.e. $DF^{-1}(F(x_0)) = [DF(x_0)]^{-1}$

Remark. When $\det DF(a) = 0$, F may also be invertible near a. For example, $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. $f'(0) = 0$, but f has inverse $f^{-1}(x) = \sqrt[3]{x}$ near 0.

Example. The conversion between polar coordinates & Cartesian coordinates for $\mathbb{R}^2$ is given by:

$$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$(r, \theta) \longmapsto (r\cos\theta, r\sin\theta)$$

$$DF = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$\det DF = r\cos^2\theta + r\sin^2\theta = r$$

We see that $\det DF \neq 0$ whenever $r \neq 0$, i.e. the point is not the origin. So the Inverse Function Theorem tells us that for any point $P \neq \vec{0}$ in $\mathbb{R}^2$, there's a neighbourhood of P on which the Cartesian coordinates are in one-to-one correspondence with the polar coordinates, i.e. two points have the same cartesian coordinates iff they have the same polar coordinates