Example. We know that the dot product is represented in suffix notation as δ_{ij}. Now if we change the coordinates by $A \in O_n(\mathbb{R})$, then the suffix notation for the dot product in the new coordinates is

$$\delta'_{i'j'} = a_{i'i} a_{j'j} \delta_{ij} = a_{i'i} a_{j'i} = \delta_{i'j'} \text{ since } A \in O_n(\mathbb{R})$$

So we see the matrix of dot product with repect to any orthonormal basis is the identity matrix I_n

Example. ε_{ijk} is the suffix notation for the 3-tensor $\det \in \Lambda^3(\mathbb{R}^3)$. If we change the coordinates by $A \in O_n(\mathbb{R})$; recall we have proved the Theorem (on Page 74), a special case is:

$$T(\vec{v}_1', \vec{v}_2', \vec{v}_3') = \det(A)\, T(\vec{v}_1, \vec{v}_2, \vec{v}_3) \text{ if } T \in \Lambda^3(\mathbb{R}^3)$$

So we see under a change of basis:

$$\varepsilon'_{i'j'k'} = \det(A)\, \varepsilon_{ijk}$$

If $\det(A) = 1$, we will get $\varepsilon'_{ijk} = \varepsilon_{ijk}$.
If $\det(A) = -1$, we will get $\varepsilon'_{ijk} = -\varepsilon_{ijk}$.

But $A \in O_n(\mathbb{R})$, these are the only two possibilities since

$$A \in O_n(\mathbb{R}) \Rightarrow A \cdot A^T = I \Rightarrow \det(A)\det(A^T) = 1$$
$$\Rightarrow \det(A) \cdot \det(A) = 1$$
$$\Rightarrow \det(A) = \pm 1$$

Definition The special orthogonal linear group is

$$SO_n(\mathbb{R}) = \{A \in O_n(\mathbb{R}) \mid \det(A) = 1\}$$

Remark. (1) Elements in $SO_n(\mathbb{R})$ corresponds to rotations in $\mathbb{R}^n$ fixing the origin.

(2) A: transition matrix. A is called orientation preserving if $\det(A) > 0$; it's called orientation reversing if $\det(A) < 0$.

(3) $\{\vec{V}_1, \dots, \vec{V}_n\}$ and $\{\vec{V}_1', \dots, \vec{V}_n'\}$ are two sets of basis for $\mathbb{R}^n$. We say they have the same orientation if the transition matrix is orientation preserving, and we say they have the opposite orientation if the transition matrix is orientation reversing.

So in the previous example, we see the alternating tensor ε_{ijk} has the same expression under the change of basis by a rotation.

Example. If $T \in T^2(V)$ such that in suffix notation with respect to some orthonormal basis $T_{ij} = T_{ji}$. Then in any other orthonormal basis, the suffix notation also satisfies $T'_{i'j'} = T'_{j'i'}$

Let $A \in O_n(\mathbb{R})$ be the change of basis matrix then $T'_{i'j'} = a_{i'i} a_{j'j} T_{ij} = a_{i'i} a_{j'j} T_{ji} = a_{j'j} a_{i'i} T_{ji} = T'_{j'i'}$

Definition. A tensor $T \in T^k(V)$ is called isotropic if its components are the same in each Cartesian coordinates

Example
- The dot product is an isotropic 2-tensor
- The determinand for $\mathbb{R}^3$ is an isotropic 3-tensor

Proposition The zero tensor is the only isotropic 1-tensor on $\mathbb{R}^3$

Proof. Suppose with repect to some coordinates, the isotopic 1-tensor is denoted by T_i.
Now Let the transformation matrix be $A = \begin{pmatrix} 0 & 1 & \\ -1 & 0 & \\ & & 1 \end{pmatrix} \in SO_3(\mathbb{R})$
Then

$$T'_j = a_{ji} T_i$$

so $T'_1 = a_{11} T_1 + a_{12} T_2 + a_{13} T_3 = T_2$

$T'_2 = a_{21} T_1 + a_{22} T_2 + a_{23} T_3 = -T_1$

If T is isotropic then $T'_1 = T_1$, $T'_2 = T_2 \Rightarrow T_1 = T_2$, $-T_1 = T_2$

$\Rightarrow T_1 = T_2 = 0$

Similarly by an appropriate choice of A, we can also show $T_3 = 0$

Proposition. If T is an isotropic 2-tensor on $\mathbb{R}^3$, then $T = \lambda\delta_{ij}$ in any Cartesian coordinates

Proof. For T an isotropic 2-tensor, if in some Cartesian coordinates it's T_{ij}, then take the transformation matrix to be

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO_3(\mathbb{R})$$

$$T'_{i'j'} = a_{i'i}a_{j'j}T_{ij} = a_{i'i}T_{ij}a_{j'j} = a_{i'i}T_{ij}a^t_{jj'} = (ATA^t)_{i'j'}$$

$$ATA^t = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} T_{22} & -T_{21} & T_{23} \\ -T_{12} & T_{11} & -T_{13} \\ T_{32} & -T_{31} & T_{33} \end{pmatrix}$$

Since T is isotropic, $T'_{i'j'} = T_{i'j'}$, so

$$T_{11} = T_{22}, \quad -T_{21} = T_{12}, \quad -T_{31} = T_{13}.$$

Similarly, we can finally show $T_{11} = T_{22} = T_{33}$, and $T_{ij} = 0$ if $i \neq j$.
Let $\lambda = T_{11}$, we thus see

$$T_{ij} = \lambda\delta_{ij}$$

Proposition. If T is an isotropic 3-tensor on $\mathbb{R}^3$, then $T = \lambda\varepsilon_{ijk}$ in any Cartesian coordinates.

Proof. For T an isotropic 3-tensor, if in some Cartesian coordinates it's T_{ijk}, then again take the transformation matrix to be $A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$T'_{i'j'k'} = a_{i'i}\, a_{j'j}\, a_{k'k}\, T_{ijk}$$

We get $T'_{123} = \sum a_{1i}\, a_{2j}\, a_{3k}\, T_{ijk} = -T_{213}$

and T is isotropic $\Rightarrow$ $T_{123} = T'_{123} = -T_{213}$

Similarly we can find the other relations to finish the proof.

Theorem If $T \in \mathcal{T}^k(\mathbb{R}^3)$ and $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, then in suffix notation
$T(\vec{u}, \vec{v}, \vec{w}) = T_{ijk}\, u_i\, v_j\, w_k$

Proof. $$T(\vec{u}, \vec{v}, \vec{w}) = T\left(\sum_{i=1}^{3} u_i \vec{e}_i, \sum_{j=1}^{3} v_j \vec{e}_j, \sum_{k=1}^{3} w_k \vec{e}_k\right)$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} u_i v_j w_k\, T(\vec{e}_i, \vec{e}_j, \vec{e}_k)$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} u_i v_j w_k\, T_{ijk}$$

So in suffix notation, $T(\vec{u}, \vec{v}, \vec{w})$ is $T_{ijk}\, u_i\, v_j\, w_k$.

Theorem (The Quotient Rule). If a suffix equation $a_i = T_{ij} b_j$ holds in all Cartesian coordinates, and for any 1-tensor b_j, a_i is also a 1-tensor. then T_{ij} represents a 2-tensor.

Proof: Let L be the change of basis matrix

$$a'_i = L_{ik} a_k = L_{ik} T_{kj} b_j$$

$$b_j = (L^{-1})_{jm} b_m \Rightarrow b_j = L_{mj} b'_m, \quad \text{since } L^{-1} = L^T$$

Then $a'_i = L_{ik} T_{kj} b_j = L_{ik} T_{kj} L_{mj} b'_m$

The assumption $a_i = T_{ij} b_j$ holds for all coordinates implies $a'_{im} = T'_{im} b'_m$

so $T'_{im} = L_{ik} T_{kj} L_{mj} = L_{ik} L_{mj} T_{kj}$ for any L.

We conclude T_{ij} is a 2-tensor