TENSORS IN CARTESIAN COORDINATES

The Cartesian Coordinates is the most popular one for the Euclidean space $\mathbb{R}^n$. Once we have fixed a cartesian coordinates (equivalently, a set of orthonormal basis), we can express an k-tensor as n^k numbers $\{a_{i_1 \dots i_k}\}_{1 \le i_1, \dots, i_k \le n}$. But, if we build another Cartesian Coordinates, we may get another set of n^k numbers $\{b_{i_1 \dots i_k}\}_{1 \le i_1, \dots, i_k \le n}$

So now we are facing the question: how will the n^k numbers change if the coordinate is changed?

Definition. If $(\vec{u}, \vec{v}) = \vec{u}^T \vec{v}$ is the dot product on $\mathbb{R}^n$, a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if for any $\vec{u}, \vec{v} \in \mathbb{R}^n$, $(T(\vec{u}), T(\vec{v})) = (\vec{u}, \vec{v})$.

Definition An $n \times n$ real matrix A is orthogonal if $A^{-1} = A^T$.

Lemma. If $\{\vec{e}_1, \dots, \vec{e}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal if and only if the matrix of T with respect to this basis is an orthogonal matrix.

Proof. If $T: \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal linear transformation, Its matrix with respect to the basis is A, then $T(\vec{u}) = A\vec{u}$, so:

$$\vec{u}^T \vec{v} = (\vec{u}, \vec{v}) = (T(\vec{u}), T(\vec{v})) = (A\vec{u})^T (A\vec{v}) = \vec{u}^T A^T A \vec{v}$$

holds for all $\vec{u}$ & $\vec{v} \in \mathbb{R}^n$. we conclude $A^T A = I_n$, the identity matrix, so $A^{-1} = A^T$

Conversely, if the matrix A of a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal matrix, then for any $\vec{u}, \vec{v} \in \mathbb{R}^n$.

$$(T(\vec{u}), T(\vec{v})) = (T(\vec{u}))^T (T(\vec{v})) = (A\vec{u})^T (A\vec{v}) = u^T A^T A v$$
$$= \vec{u}^T \vec{v}$$
$$= (\vec{u}, \vec{v})$$

So T is orthogonal linear transformation.

Definition The set of all $n \times n$ orthogonal matrices is denoted by $O_n(\mathbb{R})$, called the orthogonal linear group.

Proposition If $\{\vec{V}_1, \dots, \vec{V}_n\}$ and $\{\vec{V}_1', \dots, \vec{V}_n'\}$ are two sets of basis of $\mathbb{R}^n$ with transition matrix $A \in O_n(\mathbb{R})$, i.e. $A = (a_{ij}) \in O_n(\mathbb{R})$ and $\vec{V}_i' = \sum_{j=1}^{n} a_{ij} \vec{V}_j$. Then the coefficients of a k-tensor $T \in \mathcal{T}^k(V)$ with respect to these two basis are related in suffix notation by:

$$C'_{j_1 \cdots j_k} = a_{j_1 i_1} \cdots a_{j_k i_k} C_{i_1 \cdots i_k}$$

Remark Tensors can be expressed in suffix notation just like vectors and matrices. Fixing $\{\vec{e}_1, \dots, \vec{e}_n\}$ the cartesian coordinates, we know a k-tensor $T \in \mathcal{T}^k(\mathbb{R}^n)$ can be written as $\sum_i C_{i_1 \cdots i_k} \varphi_{i_1} \otimes \cdots \varphi_{i_k}$, so we denote it by its component $C_{i_1 \cdots i_k}$ in suffix notation

Proof. Let $\{\varphi_1, \dots, \varphi_n\}$ be the dual basis of $\{\vec{v}_1, \dots, \vec{v}_n\}$

$\{\varphi_1', \dots, \varphi_n'\}$ be the dual basis of $\{\vec{v}_1', \dots, \vec{v}_n'\}$

Then $\varphi_i(\vec{v}_k') = \varphi_i\left(\sum_{j=1}^{n} a_{kj}\vec{v}_j\right) = a_{ki}$.

we see $\varphi_i = \sum_{k=1}^{n} a_{ki}\varphi_k'$

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} = \left(\sum_{j_1=1}^{n} a_{j_1 i_1}\varphi_{j_1}'\right) \otimes \cdots \otimes \left(\sum_{j_k=1}^{n} a_{j_k i_k}\varphi_{j_k}'\right)$$

$$= \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} a_{j_1 i_1} \cdots a_{j_k i_k} \varphi_{j_1}' \otimes \cdots \otimes \varphi_{j_k}'$$

If $T \in T^k(V)$ such that $C_{i_1 \cdots i_k}$ are the coefficients with respect to $\{\varphi_1 \otimes \cdots \otimes \varphi_k\}$, and $C_{j_1 \cdots j_k}'$ are the coefficients with respect to $\{\varphi_1' \otimes \cdots \otimes \varphi_k'\}$.

Then: $T = \sum_{1 \le i_1, \dots, i_k \le n} C_{i_1 \cdots i_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$

$$= \sum_{1 \le i_1, \dots, i_k \le n} C_{i_1 \cdots i_k} \left(\sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} a_{j_1 i_1} \cdots a_{j_k i_k}\right) \varphi_{j_1}' \otimes \cdots \otimes \varphi_{j_k}'$$

$$= \sum_{1 \le j_1, \dots, j_k \le n} \sum_{1 \le i_1, \dots, i_k \le n} C_{i_1 \cdots i_k} a_{j_1 i_1} \cdots a_{j_k i_k} \varphi_{j_1}' \otimes \cdots \otimes \varphi_{j_k}'$$

So we get $C_{j_1 \cdots j_k}' = \sum_{1 \le i_1, \dots, i_k \le n} a_{j_1 i_1} \cdots a_{j_k i_k} C_{i_1 \cdots i_k}$

In suffix notation:

$$C_{j_1 \cdots j_k}' = a_{j_1 i_1} \cdots a_{j_k i_k} C_{i_1 \cdots i_k}$$