Theorem. If V has basis $\{\vec{v}_1, \dots, \vec{v}_n\}$ and V^* has dual basis $\{\varphi_1, \dots, \varphi_n\}$, then $\{\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}\}$, $1 \le i_1 < i_2 < \dots < i_k \le n$ forms a basis for $\Lambda^k(V)$. So $\dim \Lambda^k(V) = \frac{n!}{k!(n-k)!}$

Proof. If $\omega \in \Lambda^k(V)$, we can write it as

$$\omega = \sum_{i_1, \dots, i_k} a_{i_1 \dots i_k} \varphi_{i_1} \otimes \dots \otimes \varphi_{i_k}$$

$$\omega = \mathrm{Alt}(\omega) = \sum_{i_1, \dots, i_k} a_{i_1 \dots i_k} \mathrm{Alt}(\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k})$$

and each $\mathrm{Alt}(\varphi_{i_1} \otimes \dots \otimes \varphi_{i_k}) = \begin{cases} \frac{1}{k!} \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} & \text{if } i_1, \dots, i_k \text{ distinct} \\ 0 & \text{otherwise} \end{cases}$

so $\{\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}\}$ span $\Lambda^k(V)$.

We also need to show the linearly independence:

If $\sum c_{i_1, \dots, i_k} \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k} = 0$.

Observe $\varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}(\vec{v}_{j_1}, \dots, \vec{v}_{j_k}) = \begin{cases} 1 & \text{if } i_1 = j_1, \dots, i_k = j_k \\ 0 & \text{otherwise} \end{cases}$

(Exercise)

$(j_1 < \dots < j_k)$.

So we can get linearly independence by applying

$\sum c_{i_1, \dots, i_k} \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$ to each $(\vec{v}_{j_1}, \dots, \vec{v}_{j_k})$

Corollary If $\dim V = n$, then $\dim \Lambda^n(V) = 1$

Theorem. Let $\vec{v}_1, \ldots, \vec{v}_n$ be a basis for $\mathbb{R}^n$. $T \in \Lambda^n(\mathbb{R}^n)$.
If $\vec{w}_i = \sum_{j=1}^{n} a_{ij} \vec{v}_j$ are n vectors in V, then:

$$T(\vec{w}_1, \ldots, \vec{w}_n) = \det(a_{ij}) \cdot T(\vec{v}_1, \ldots, \vec{v}_n)$$

Proof Define $S \in \Lambda^n(\mathbb{R}^n)$ by

$$S\left(\sum_{j=1}^{n} a_{1j} \vec{e}_j, \ldots, \sum_{j=1}^{n} a_{nj} \vec{e}_j\right) = T\left(\sum_{j=1}^{n} a_{1j} \vec{v}_1, \ldots, \sum_{j=1}^{n} a_{nj} \vec{v}_j\right)$$

where $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is the standard basis for $\mathbb{R}^n$.

Since $\det \in \Lambda^n(\mathbb{R}^n)$ is nonzero, $\dim \Lambda^n(\mathbb{R}^n) = 1$,
so $\exists\, \lambda \in \mathbb{R}$ such that

$$S = \lambda \det$$

Then
$$\begin{aligned} T(\vec{w}_1, \ldots, \vec{w}_n) &= S\left(\sum_{j=1}^{n} a_{1j} \vec{e}_j, \ldots, \sum_{j=1}^{n} a_{nj} \vec{e}_j\right) \\ &= \lambda \det\left(\sum_{j=1}^{n} a_{1j} \vec{e}_j, \ldots, \sum_{j=1}^{n} a_{nj} \vec{e}_j\right) \\ &= \lambda \det(a_{ij}) \end{aligned}$$

$$T(\vec{v}_1, \ldots, \vec{v}_n) = S(\vec{e}_1, \ldots, \vec{e}_n) = \lambda \det(\delta_{ij}) = \lambda$$

So $T(\vec{w}_1, \ldots, \vec{w}_n) = \det(a_{ij})\, T(\vec{v}_1, \ldots, \vec{v}_n)$

Corollary. Take the standard basis $\{\vec{e}_1, \dots, \vec{e}_n\}$ of $\mathbb{R}^n$. $\{\delta_1, \dots, \delta_n\}$ is its dual basis on V^*

Then $\det = \delta_1 \wedge \delta_2 \wedge \dots \wedge \delta_n$

Remark. By the discussion above, we have another way to define det:

The determinant map for $\mathbb{R}^n$ is the unique alternating n-tensor whose value on $(\vec{e}_1, \dots, \vec{e}_n)$ is 1. where $\{\vec{e}_1, \dots, \vec{e}_n\}$ is the standard basis for $\mathbb{R}^n$.

Example. Recall when we studies suffix notations, we defined the "alternating tensor" ε_{ijk}.

If we take the standard basis $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ on $\mathbb{R}^3$. $\{\varphi_1, \varphi_2, \varphi_3\}$ is the dual basis. Then we have shown before

$$\varepsilon_{ijk} u_i v_j w_k = \vec{u} \cdot (\vec{v} \times \vec{w}) = \det(\vec{u}, \vec{v}, \vec{w})$$

So ε_{ijk} indeed represents the determinant on $\mathbb{R}^3$, and this expression above tells us that

$$\det = \sum_{0 \le i,j,k \le 3} \varepsilon_{ijk} \varphi_i \otimes \varphi_j \otimes \varphi_k$$

DIFFERENTIAL FORMS (Brief Introduction)

Recall that given a smooth function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$, at $p \in \mathbb{R}^n$, we can compute the directional derivative along the direction $\vec{V} = (v_1, \ldots, v_n)$:

$$D_{\vec{V}}(f) = \vec{V} \cdot \vec{\nabla} f = v_1 \frac{\partial f}{\partial x_1}(p) + \cdots + v_n \frac{\partial f}{\partial x_n}(p)$$

Since this holds for all smooth functions, we can define the operator

$$D_{\vec{V}} = \vec{V} \cdot \vec{\nabla} = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}$$

This expression indicates we can use a new notation for the standard basis of $\mathbb{R}^n$: $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$.

i.e. $\frac{\partial}{\partial x_i} = (0, \ldots, \underset{\substack{\uparrow \\ i\text{th}}}{1}, \ldots 0)$

Given a smooth function $f: \mathbb{R}^n \to \mathbb{R}$, there is a map

$$\begin{aligned} \mathbb{R}^n &\xrightarrow{df} (\mathbb{R}^n)^* \\ p &\longmapsto df(p) \end{aligned} \qquad (\mathbb{R}^n)^* \text{ is the dual space of } \mathbb{R}^n$$

where $df(p)$ acts on $v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n} \in \mathbb{R}^n$ by

$$df(p)\left(v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}\right) = v_1 \frac{\partial f}{\partial x_1}(p) + \cdots + v_n \frac{\partial f}{\partial x_n}(p)$$

Let x_i be the function $x_i: \mathbb{R}^n \longrightarrow \mathbb{R}$

$$(p_1, \ldots, p_n) \longmapsto p_i.$$

Then $dx_i(p)(\frac{\partial}{\partial x_j}) = \frac{\partial x_i}{\partial x_j}(p) = \delta_{ij} \quad \forall p \in \mathbb{R}^n$

so $\{dx_1(p), \dots, dx_n(p)\}$ forms the dual basis of $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ for any $p \in \mathbb{R}^n$

This implies for each smooth function $f: \mathbb{R}^n \to \mathbb{R}$

$$df(p) = \frac{\partial f}{\partial x_1}(p)\, dx_1(p) + \cdots + \frac{\partial f}{\partial x_n}(p)\, dx_n(p) \quad \forall p \in \mathbb{R}^n$$

So $\boxed{df = \frac{\partial f}{\partial x_1} dx_1 + \cdots + \frac{\partial f}{\partial x_n} dx_n}$, called the differential of f

In other words, the differential of f tells us how to write df as a linear combination of the basis $\{dx_1, \dots, dx_n\}$ at each point.

Definition A differential k-form on $\mathbb{R}^n$ is a map $\omega: \mathbb{R}^n \to \Lambda^k(\mathbb{R}^n)$

Example df is a differential 1-form.

A differential k-form ω can be written as

$$\omega(p) = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \cdots i_k}(p)\, dx_{i_1}(p) \wedge \cdots dx_{i_k}(p)$$

i.e. $\omega = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1 \cdots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

Definition ω is a continuous/differentiable/smooth k-form if all the $\omega_{i_1 \cdots i_k}$ are continuous/differentiable/smooth functions.