Proposition. $T \in T^k(V)$. T is alternating if and only if
$T(\vec{w}_1, \dots, \vec{w}_k) = 0$ whenever some $\vec{w}_i = \vec{w}_j$ for $i \neq j$

Proof. If T is an alternating tensor, $(\vec{w}_1, \dots, \vec{w}_k) \in V \times \dots \times V$ such that $\vec{w}_i = \vec{w}_j$ for some $i \neq j$, then

$$T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_j, \dots, \vec{w}_k) = -T(\vec{w}_1, \dots, \vec{w}_j, \dots, \vec{w}_i, \dots, \vec{w}_k)$$

$$= -T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_j, \dots, \vec{w}_k)$$

so $T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_j, \dots, \vec{w}_k) = 0$

Conversely, if for any $(\vec{w}_1, \dots, \vec{w}_k) \in V \times \dots \times V$ such that $\vec{w}_i = \vec{w}_j$ for some $i \neq j$, $T(\vec{w}_1, \dots, \vec{w}_k) = 0$.

then

$$T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_j, \dots, \vec{w}_n)$$
$$= T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_j, \dots, \vec{w}_n) + T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_i, \dots, \vec{w}_n)$$
$$= T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_i + \vec{w}_j, \dots, \vec{w}_n)$$

Similarly, $T(\vec{w}_1, \dots, \vec{w}_j, \dots, \vec{w}_i, \dots, \vec{w}_n)$
$$= T(\vec{w}_1, \dots, \vec{w}_j, \dots, \vec{w}_i + \vec{w}_j, \dots, \vec{w}_n)$$

So $T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_j, \dots, \vec{w}_n) + T(\vec{w}_1, \dots, \vec{w}_j, \dots, \vec{w}_i, \dots, \vec{w}_n)$
$$= T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_i + \vec{w}_j, \dots, \vec{w}_n) + T(\vec{w}_1, \dots, \vec{w}_j, \dots, \vec{w}_i + \vec{w}_j, \dots, \vec{w}_n)$$
$$= T(\vec{w}_1, \dots, \vec{w}_i + \vec{w}_j, \dots, \vec{w}_i + \vec{w}_j, \dots, \vec{w}_n)$$
$$= 0$$

Corollary. If $T \in T^k(V)$ is an alternating tensor, $\{\vec{w}_1, \dots, \vec{w}_k\}$ is a set of k linearly depent vectors in V. then

$$T(\vec{w}_1, \dots, \vec{w}_k) = 0$$

Proof. Will be homework.

Given any $T \in T^k(V)$, there is a way to construct an alternating k-tensor, $Alt(T)$, as follows:

$$Alt(T)(\vec{v}_1, \dots, \vec{v}_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn\, \sigma \cdot T(\vec{v}_{\sigma(1)}, \dots, \vec{v}_{\sigma(k)})$$

where $\sigma \in S_k$ is a permutation of k letters, and $sgn(\sigma)$ is the signature of σ. i.e. if we decompose σ as a product of m transpositions, $sgn(\sigma) = (-1)^m$

Definition. The set of all alternating k tensors form a subspace of $T^k(V)$, and it's denoted as $\Lambda^k(V)$.

Theorem. (i) If $T \in T^k(V)$, then $AlT(T) \in \Lambda^k(V)$
(ii) If $T \in \Lambda^k(V)$, then $AlT(T) = T$

Proof. $AlT(T)(\vec{v}_1, \dots, \vec{v}_i, \dots, \vec{v}_j, \dots, \vec{v}_k)$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} sgn\, \sigma \cdot T(\vec{v}_{\sigma(1)}, \dots, \vec{v}_{\sigma(i)}, \dots, \vec{v}_{\sigma(j)}, \dots, \vec{v}_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma(ij) \in S_k} sgn((ij)\sigma)\, T(\vec{v}_{\sigma(1)}, \dots, \vec{v}_{\sigma(j)}, \dots, \vec{v}_{\sigma(i)}, \dots, \vec{v}_{\sigma(k)})$$

$$= -\frac{1}{k!} \sum_{\sigma \in S_k} sgn\, \sigma \cdot T(\vec{v}_{\sigma(1)}, \dots, \vec{v}_{\sigma(j)}, \dots, \vec{v}_{\sigma(i)}, \dots, \vec{v}_{\sigma(k)})$$

$$= -AlT(T)(\vec{v}_1, \dots, \vec{v}_j, \dots, \vec{v}_i, \dots, \vec{v}_k)$$

(ii). If $T \in \Lambda^k(V)$,

$$\mathrm{Alt}(T)(\vec{v}_1, \dots, \vec{v}_k)$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \cdot T(\vec{v}_{\sigma(1)}, \dots, \vec{v}_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}\sigma \cdot \operatorname{sgn}\sigma\, T(\vec{v}_1, \dots, \vec{v}_k)$$

$$= \frac{1}{k!} \cdot k! \cdot T(\vec{v}_1, \dots, \vec{v}_k)$$

$$= T(\vec{v}_1, \dots, \vec{v}_k)$$

Exercise. In the above proof, we made use of the fact:
$T \in \Lambda^k(V)$, $T(\vec{v}_{\sigma(1)}, \dots, \vec{v}_{\sigma(k)}) = \operatorname{sgn}(\sigma)\, T(\vec{v}_1, \dots, \vec{v}_k)$
Prove this fact.

Definition. The wedge product $\Lambda^k(V) \times \Lambda^l(V) \xrightarrow{\wedge} \Lambda^{k+l}(V)$
is defined as follows: if $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$,
then $\omega \wedge \eta = \frac{(k+l)!}{k!\,l!} \mathrm{Alt}(\omega \otimes \eta)$

Proposition. The following properties hold for wedge product:

(i). $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$

(ii). $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$

(iii) $a\omega \wedge \eta = \omega \wedge (a\eta) = a(\omega \wedge \eta)$ $\quad (a \in \mathbb{F})$

(iv). $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$, if $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$

(v). $(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) = \frac{(k+l+m)!}{k!\,l!\,m!} \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$
if $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, $\theta \in \Lambda^m(V)$.

(vi) $\omega_1 \wedge \dots \wedge \omega_k = \frac{(l_1 + \dots + l_k)!}{l_1! \cdots l_k!} \mathrm{Alt}(\omega_1 \otimes \dots \otimes \omega_k)$

if $\omega_1 \in \Lambda^{l_1}(V)$, ..., $\omega_k \in \Lambda^{l_k}(V)$