Definition. A bilinear form on a vector space V is positive definite if $\forall \vec{v} \in V$ and $\vec{v} \neq \vec{0}$, $\langle \vec{v}, \vec{v} \rangle > 0$

Theorem. (Gram-Schmidt) If $\langle \; , \; \rangle$ is a symmetric, positive definite bilinear form on V, then there exists an orthonormal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ of V with respect to $\langle \; , \; \rangle$. i.e. $\langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij}$

Proof. Let $\{\vec{w}_1, \ldots, \vec{w}_n\}$ be a basis of V.

Let
$$\vec{w}_1' = \vec{w}_1$$
$$\vec{w}_2' = \vec{w}_2 - \frac{\langle \vec{w}_1', \vec{w}_2 \rangle}{\langle \vec{w}_1', \vec{w}_1' \rangle} \vec{w}_1'$$
$$\vec{w}_3' = \vec{w}_3 - \frac{\langle \vec{w}_1', \vec{w}_3 \rangle}{\langle \vec{w}_1', \vec{w}_1' \rangle} \vec{w}_1' - \frac{\langle \vec{w}_2', \vec{w}_3 \rangle}{\langle \vec{w}_2', \vec{w}_2' \rangle} \vec{w}_2'$$
$$\vdots$$
$$\vec{w}_n' = \vec{w}_n - \frac{\langle \vec{w}_1', \vec{w}_n \rangle}{\langle \vec{w}_1', \vec{w}_1' \rangle} \vec{w}_1' - \cdots - \frac{\langle \vec{w}_{n-1}', \vec{w}_n \rangle}{\langle \vec{w}_{n-1}', \vec{w}_{n-1}' \rangle} \vec{w}_{n-1}'$$

We can show that if $i \neq j$, $\langle \vec{w}_i', \vec{w}_j' \rangle = 0$ by induction.

First,
$$\langle \vec{w}_1', \vec{w}_2' \rangle = \langle \vec{w}_1, \vec{w}_2 \rangle - \frac{\langle \vec{w}_1', \vec{w}_2 \rangle}{\langle \vec{w}_1', \vec{w}_1' \rangle} \langle \vec{w}_1, \vec{w}_1' \rangle$$
$$= \langle \vec{w}_1, \vec{w}_2 \rangle - \langle \vec{w}_1, \vec{w}_2 \rangle$$
$$= 0$$

Next assume for $1 \leq i < j \leq k$, $\langle \vec{w}_i', \vec{w}_j' \rangle = 0$.
then for $k+1$,
$$\langle \vec{w}_{k+1}', \vec{w}_i' \rangle = \left\langle \vec{w}_{k+1} - \frac{\langle \vec{w}_1', \vec{w}_{k+1} \rangle}{\langle \vec{w}_1', \vec{w}_1' \rangle} \vec{w}_1' - \cdots - \frac{\langle \vec{w}_k', \vec{w}_{k+1} \rangle}{\langle \vec{w}_k', \vec{w}_k' \rangle} \vec{w}_k', \vec{w}_i' \right\rangle$$
$$= \langle \vec{w}_{k+1}, \vec{w}_i' \rangle - \frac{\langle \vec{w}_i', \vec{w}_{k+1} \rangle}{\langle \vec{w}_i', \vec{w}_i' \rangle} \cdot \langle \vec{w}_i', \vec{w}_i' \rangle = 0$$

At last, we do a "normalization":

Let $\vec{v}_i = \dfrac{\vec{w}_i'}{\sqrt{\langle \vec{w}_i', \vec{w}_i' \rangle}}$ for $i = 1, \dots, n$.

$$\langle \vec{v}_i, \vec{v}_i \rangle = \frac{\langle \vec{w}_i', \vec{w}_i' \rangle}{\langle \vec{w}_i', \vec{w}_i' \rangle} = 1$$

$$\langle \vec{v}_i, \vec{v}_j \rangle = \frac{\langle \vec{w}_i', \vec{w}_j' \rangle}{\sqrt{\langle \vec{w}_i', \vec{w}_i' \rangle} \cdot \sqrt{\langle \vec{w}_j', \vec{w}_j' \rangle}} = 0 \quad \text{if } i \neq j$$

so $\langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij}$.

And this also implies $\vec{v}_1, \dots, \vec{v}_n$ are linearly independent:
if $a_1\vec{v}_1 + \dots + a_n\vec{v}_n = \vec{0}$, then for any $\vec{v}_i$,

$$0 = \langle a_1\vec{v}_1 + \dots + a_n\vec{v}_n, \vec{v}_i \rangle = a_i \langle \vec{v}_i, \vec{v}_i \rangle = a_i$$

and we know V is of dimension n, so $\{\vec{v}_1, \dots, \vec{v}_n\}$ form a basis of V.

Example. The determinant of $n \times n$ matrices can be interpreted as an n-tensor on a vector space $\mathbb{R}^n$:
Let $\{\vec{e}_1, \dots, \vec{e}_n\}$ be the standard basis of $\mathbb{R}^n$

$$\mathbb{R}^n \times \dots \times \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$\left(\sum_{j=1}^{n} a_{1j}\vec{e}_j, \dots, \sum_{j=1}^{n} a_{nj}\vec{e}_j\right) \longmapsto \det(a_{ij})$$

Definition. $T \in T^k(V)$ is alternating if for any $1 \leq i < j \leq k$,
$$T(\vec{w}_1, \dots, \vec{w}_i, \dots, \vec{w}_j, \dots, \vec{w}_k) = -T(\vec{w}_1, \dots, \vec{w}_j, \dots, \vec{w}_i, \dots, \vec{w}_k)$$