TENSORS AS MULTILINEAR MAPS

Definition. V is a vector space over a field $\mathbb{F}$. m is a positive integer. Define an m-tensor on V to be a multilinear map $\underbrace{V \times V \times \cdots \times V}_{m \text{ copies}} \xrightarrow{T} \mathbb{F}$

i.e. for each $1 \leq i \leq m$,

$$T(\vec{v}_1, \ldots, \lambda \vec{v}_i + \mu \vec{v}_i{}', \ldots, \vec{v}_n) = \lambda T(\vec{v}_1, \ldots, \vec{v}_i, \ldots \vec{v}_n) + \mu T(\vec{v}_1, \ldots \vec{v}_i{}', \ldots \vec{v}_n)$$

Definition. The set of m-tensors on V is denoted as $T^m(V)$.

Proposition. $T^m(V)$ is a $\mathbb{F}$-vector space, with addition:

$$(S+T)(\vec{v}_1, \ldots, \vec{v}_n) = S(\vec{v}_1, \ldots, \vec{v}_n) + T(\vec{v}_1, \ldots, \vec{v}_n)$$

Scalar multiplication: $(\lambda S)(\vec{v}_1, \ldots, \vec{v}_n) = \lambda S(\vec{v}_1, \ldots, \vec{v}_n)$

Proof. Exercise

Definition. Given $S \in T^k(V)$ and $T \in T^l(V)$, define the tensor product $S \otimes T \in T^{k+l}(V)$ to be the multilinear map

$$S \otimes T(\vec{v}_1, \ldots, \vec{v}_k, \vec{v}_{k+1}, \ldots, \vec{v}_{k+l}) = S(\vec{v}_1, \ldots, \vec{v}_k) \cdot T(\vec{v}_{k+1}, \ldots, \vec{v}_{k+l})$$

Proposition. The following rules hold for tensors.

$$(S_1 + S_2) \otimes T = S_1 \otimes T + S_2 \otimes T$$

$$T \otimes (S_1 + S_2) = T \otimes S_1 + T \otimes S_2$$

$$(aS) \otimes T = S \otimes (aT) = a(S \otimes T)$$

$$(S \otimes T) \otimes U = S \otimes (T \otimes U)$$

Remark · Note in general, $S \otimes T \neq T \otimes S$.

· Since tensor product is associative, we can write $S \otimes T \otimes U$ to denote either $(S \otimes T) \otimes U$ or $S \otimes (T \otimes U)$

Theorem. Let $\{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis of V, and $\{\varphi_1, \ldots, \varphi_n\}$ be the dual basis, (i.e. $\varphi_i(\vec{v}_j) = \delta_{ij}$). Then the set of all k-fold tensor products

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \quad 1 \le i_1, \ldots, i_k \le n$$

is a basis for $T^k(V)$, so $\dim T^k(V) = n^k$

Proof. First we prove these n^k k-tensor products span $T^k(V)$:

$$\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(\vec{v}_{j_1}, \ldots, \vec{v}_{j_k}) = \varphi_{i_1}(\vec{v}_{j_1}) \cdot \ldots \cdot \varphi_{i_k}(\vec{v}_{j_k})$$

$$= \delta_{i_1 j_1} \cdots \delta_{i_k j_k}$$

$$= \begin{cases} 1, & \text{if } i_1 = j_1, \ldots, i_k = j_k \\ 0, & \text{otherwise} \end{cases}$$

Now for $\vec{w}_1, \ldots, \vec{w}_k$ in V, since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis of V,

$$\vec{w}_i = \sum_{j=1}^{n} a_{ij} \vec{v}_j$$

For any $T \in T^k(V)$,

$$T(\vec{w}_1, \ldots, \vec{w}_k) = T\left(\sum_{j_1=1}^{n} a_{1j_1} \vec{v}_{j_1}, \ldots, \sum_{j_k=1}^{n} a_{kj_k} \vec{v}_{j_k}\right)$$

$$= \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} a_{1j_1} \cdots a_{kj_k} T(\vec{v}_{j_1}, \ldots, \vec{v}_{j_k})$$

$$= \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} T(\vec{v}_{j_1}, \ldots, \vec{v}_{j_k}) \cdot \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}(\vec{w}_1, \ldots, \vec{w}_k)$$

So $T = \sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} T(\vec{v}_{j_1}, \dots, \vec{v}_{j_k}) \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}$

i.e. $T \in \text{Span}\{\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}\}$, $1 \le j_1, \dots, j_k \le n$

Next we prove $\{\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}\}$ are linearly independent.

If $\exists\, a_{i_1, \dots, i_k} \in \mathbb{F}$, $1 \le i_1, \dots, i_k \le n$ such that

$\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} a_{i_1, \dots, i_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}$ is the zero map.

Then $\sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} a_{i_1, \dots, i_k} \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} (\vec{v}_{j_1}, \dots, \vec{v}_{j_k}) = 0$

i.e. $a_{j_1, \dots, j_k} = 0$

We conclude $\{\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k}\}$ are linearly independent.

So we conclude $\{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}\}$ $1 \le i_1, \dots, i_k \le n$ form a basis of $\mathcal{T}^k(V)$, and $\dim \mathcal{T}^k(V) = n^k$

Example. A bilinear form $\langle \, , \rangle$ on V is a 2-tensor on V. If $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for V, and A is the $n \times n$ matrix representing $\langle \, , \rangle$ under this basis, then $\langle \, , \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} \delta_i \otimes \delta_j$, where $\delta_1, \dots, \delta_n$ is the dual basis in V^*, and $A_{ij} = \langle \vec{v}_i, \vec{v}_j \rangle$