Homework III Solution

First-Half

1. $f'(x) = -10(5x+3)^{-3}$, so $f'(0) = -\frac{10}{27}$. $f(0) = \frac{1}{9}$.
   The linear approximation around $x = 0$ is:
   $$f(x) \approx \frac{1}{9} - \frac{10}{27}x$$
2. Let $f(x) = \sqrt{1+x}$, then $f'(x) = \frac{1}{2\sqrt{1+x}}$. $f(0) = 1$ and $f'(0) = \frac{1}{2}$.
   The linear approximation gives that near $x = 0$:
   $$f(x) \approx 1 + \frac{1}{2}x$$
3. $$d(x^p + e^x) = (x^p + e^x)'dx = (px^{p-1} + e^x)dx$$
4. $f(x) = x^{\frac{1}{3}}$, so $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. $f(1) = 1$, $f'(1) = \frac{1}{3}$, so near $x = 1$, the function has linear approximation
   $$f(x) \approx 1 + \frac{1}{3}(x-1)$$
   Take $x = 1.1$, we get
   $$f(1.1) \approx 1 + \frac{1}{3}(1.1-1) = \frac{31}{30} \approx 1.033$$
5. $f'(x) = -2xe^{-x^2}$, $f''(x) = -2(1-2x^2e^{-x^2})$, so $f(0) = 1$, $f'(0) = 0$, $f''(0) = -2$
   The quadratic approximation is
   $$f(x) \approx 1 + 0(x-0) + \frac{-2}{2!}(x-0)^2 = 1 - x^2$$

6. $f'(x) = 2ax + b$, $f''(x) = 2a$, so $f(0) = c$, $f'(0) = b$, $f''(0) = 2a$.

The quadratic approximation is

$$f(x) \approx c + b(x-0) + \frac{2a}{2!}(x-0)^2 = ax^2 + bx + c$$

Second-Half

1. (a). Since $\lim_{x\to 0}(e^{-3x} - e^{-2x} + x) = e^0 - e^0 + 0 = 0$ and $\lim_{x\to 0} x^2 = 0$, we can apply the L'Hospital's Rule:

$$\lim_{x\to 0} \frac{e^{-3x} - e^{-2x} + x}{x^2} = \lim_{x\to 0} \frac{-3e^{-3x} + 2e^{-2x} + 1}{2x}$$

and $\lim_{x\to 0} -3e^{-3x} + 2e^{-2x} + 1 = -3e^0 + 2e^0 + 1 = -3 + 2 + 1 = 0$, $\lim_{x\to 0} 2x = 0$, so we can apply the L'Hospital's Rule again:

$$\lim_{x\to 0} \frac{-3e^{-3x} + 2e^{-2x} + 1}{2x} = \lim_{x\to 0} \frac{9e^{-3x} - 4e^{-2x}}{2} = \frac{5}{2}$$

(b).

$$\begin{aligned}
\lim_{x\to +\infty} \frac{x^4 - 4x^3 + 6x^2 - 8x + 8}{x^3 - 3x^2 + 4} &= \lim_{x\to +\infty} \frac{4x^3 - 12x^2 + 12x - 8}{3x^2 - 6x} \\
&= \lim_{x\to +\infty} \frac{12x^2 - 24x + 12}{6x - 6} \\
&= \lim_{x\to +\infty} \frac{24x - 24}{6} \\
&= \lim_{x\to +\infty} 4(x-1) \\
&= +\infty
\end{aligned}$$

(c). $\lim_{x\to +\infty} x^{-\frac{1}{2}} \ln x = \lim_{x\to +\infty} \frac{\ln x}{\sqrt{x}}$.

$\lim_{x\to +\infty} \ln x = +\infty$ and $\lim_{x\to +\infty} \sqrt{x} = +\infty$, so we can apply the L'Hospital's Rule:

$$\lim_{x\to +\infty} \frac{\ln x}{\sqrt{x}} = \lim_{x\to +\infty} \frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to +\infty} \frac{2}{\sqrt{x}} = 0$$

2. (a). $f(x) = 2x^{-\frac{3}{2}}$, so $f'(x) = -3x^{-\frac{5}{2}}$

$$El_x f(x) = \frac{x}{f(x)} f'(x) = \frac{x}{2x^{-\frac{3}{2}}}(-3x^{-\frac{5}{2}}) = -\frac{3}{2}$$

(b). $f(x) = -100x^{100}$, $f'(x) = -10000x^{99}$

$$El_x f(x) = \frac{x}{f(x)} f'(x) = \frac{x}{-100x^{100}}(-10000x^{99}) = 100$$

3.

$$\begin{aligned} El_x(fg) &= \frac{x}{f(x)g(x)}(f(x)g(x))' \\ &= \frac{x}{f(x)g(x)}(f'(x)g(x) + f(x)g'(x)) \\ &= \frac{x}{f(x)g(x)} f'(x)g(x) + \frac{x}{f(x)g(x)} f(x)g'(x) \\ &= \frac{x}{f(x)} f'(x) + \frac{x}{g(x)} g'(x) \\ &= El_x f + El_x g \end{aligned}$$

4.

$$-0.4 \times 10\% = -4\%$$

So the consequence of a 10% increase in fares is that the volume of passenger demand will decrease by about 4%

5. $D = Ar^{1.23}$, $D' = 1.23Ar^{0.23}$

$$El_r D = \frac{r}{D} D' = \frac{r}{Ar^{1.23}}(1.23Ar^{0.23}) = 1.23$$

The interpretation is that a 1% increase in income will lead to about 1.23% increase of demand for apples