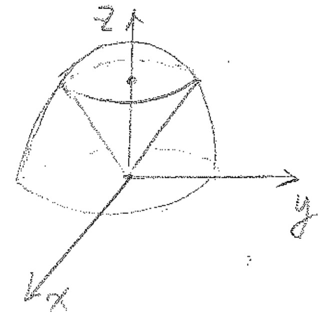
HOMEWORK V SOLUTION.

First-Half.

1. Cylindrical coordinates:



$$\iiint_E 1\,dV = \int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}}\int_r^{\sqrt{1-r^2}} 1\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\sqrt{2}}{2}} (\sqrt{1-r^2}-r)r\,dr\,d\theta$$

$$= \int_0^{2\pi} 1\,d\theta \cdot \int_0^{\frac{\sqrt{2}}{2}} (\sqrt{1-r^2}-r)\,r\,dr$$

$$= 2\pi\cdot \frac{1}{6}(2-\sqrt{2})$$

$$= \frac{(2-\sqrt{2})\pi}{3}$$

Spherical Coordinates:

$$\iiint_E 1\,dV = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^1 \rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \left(\int_0^{\frac{\pi}{4}}\sin\phi\,d\phi\right)\left(\int_0^{2\pi}1\,d\theta\right)\left(\int_0^1\rho^2\,d\rho\right)$$

$$= \left(1-\frac{\sqrt{2}}{2}\right)\cdot 2\pi\cdot\frac{1}{3}$$

$$= \frac{(2-\sqrt{2})\pi}{3}$$

2. $$\iiint_E (x+y+z)\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2}(r\cos\theta + r\sin\theta + z)\,r\,dz\,dr\,d\theta$$

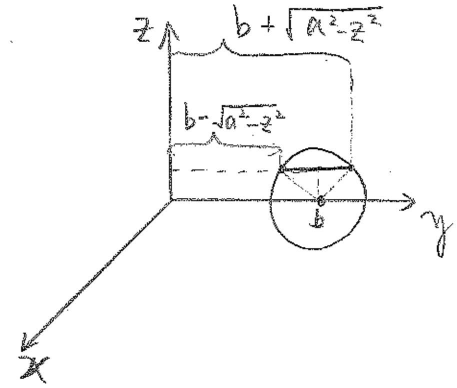
$$= \int_0^{2\pi}\int_0^2\int_0^{4-r^2} r^2\cos\theta + r^2\sin\theta + rz\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^2(4-r^2)(\cos\theta+\sin\theta) + \frac{r}{2}(4-r^2)^2\,dr\,d\theta$$

$$= \left(\int_0^{2\pi}(\cos\theta+\sin\theta)\,d\theta\right)\cdot\left(\int_0^2 r^2(4-r^2)\,dr\right) + \left(\int_0^{2\pi}\frac{1}{2}\,d\theta\right)\cdot\left(\int_0^2 r(4-r^2)^2\,dr\right)$$

$$= 0 + \pi\cdot\frac{32}{3} = \frac{32}{3}\pi$$

3. $$\iiint_E 1\,dV = \int_{-a}^{a}\int_0^{2\pi}\int_{b-\sqrt{a^2-z^2}}^{b+\sqrt{a^2-z^2}} r\,dr\,d\theta\,dz$$



$$= \int_{-a}^{a}\int_0^{2\pi} \frac{r^2}{2}\Bigg|_{b-\sqrt{a^2-z^2}}^{b+\sqrt{a^2-z^2}} d\theta\,dz$$

$$= \int_{-a}^{a}\int_0^{2\pi} 2b\sqrt{a^2-z^2}\,d\theta\,dz$$

$$= \int_{-a}^{a}\sqrt{a^2-z^2}\,dz \int_0^{2\pi} 2b\,d\theta$$

$$= \frac{\pi a^2}{2}\cdot 4b\pi$$

$$= 2\pi^2 a^2 b$$

4. C can be parameterized to be $\vec{r}(t) = \langle 4\cos t, 4\sin t\rangle$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

So $$\int_C xy^4\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4\cdot\cos t\,(4\sin t)^4\cdot|\vec{r}'(t)|\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdot 4^5\cos t\,\sin^4 t\cdot 4\,dt$$

$$= 4^6\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\sin^4 t\,dt$$

$$= 4096\cdot\frac{2}{5}$$

$$= \frac{8192}{5}$$

5. $\vec{r}(t) = \langle t, t^2, t^3 \rangle \quad , 0 \le t \le 1$

$\vec{r}'(t) = \langle 1, 2t, 3t^2 \rangle$

$$\int_C x^2+y^2+z^2\, ds = \int_0^1 \left((t)^2 + (t^2)^2 + (t^3)^2\right) \cdot |\vec{r}'(t)|\, dt$$

$$= \int_0^1 (t^2+t^4+t^6)\sqrt{1+4t^2+9t^4}\, dt$$

6. C can be parameterized by $\vec{r}(t) = \langle 3t+1, t, 2t \rangle$, $0 \le t \le 1$

So $\int_C z^2dx + x^2dy + y^2dz = \int_0^1 (2t)^2 \cdot 3dt + \int_0^1 (3t+1)^2 \cdot dt + \int_0^1 t^2 \cdot 2dt$

$$= 12\int_0^1 t^2 dt + \int_0^1 (3t+1)^2 dt + 2\int_0^1 t^2 dt$$

$$= 4 + 7 + \frac{2}{3}$$

$$= \frac{35}{3}$$

7. Let C_1 be the line segment from $(0,0,0)$ to $(1,0,1)$

C_2 be the line segment from $(1,0,1)$ to $(0,1,2)$

Then C_1 can be parameterized by $\vec{r}(t) = \langle t, 0, t \rangle$, $0 \le t \le 1$

$$\int_{C_1} (y+z)dx + (x+z)dy + (x+y)dz = \int_0^1 t \cdot dt + t\, dt = \int_0^1 2t\, dt = 1$$

C_2 can be parameterized by $\vec{r}(t) = \langle 1-t, t, 1+t \rangle$, $0 \le t \le 1$

$$\int_{C_2} (y+z)dx + (x+z)dy + (x+y)dz = \int_0^1 (t+1+t)(-1)dt + (1-t+1+t)\,dt + (1-t+t)\,dt$$

$$= \int_0^1 -(2t+1) + 2 + 1\, dt$$

$$= 1$$

So $\int_C (y+z)dx + (x+z)dy + (x+y)dz = 1+1 = 2$

8. $\int_C \vec{F}(x,y)\cdot d\vec{r} = \int_C x\,dx + (y+2)\,dy$

$= \int_0^{2\pi} (t-\sin t)\cdot(1-\cos t)\,dt + (1-\cos t+2)\cdot \sin t\,dt$

$= \int_0^{2\pi} t - \sin t - t\cos t + \sin t\cos t + 3\sin t - \sin t\cos t\ dt$

$= \int_0^{2\pi} t + 2\sin t - t\cos t\ dt$

$= 2\pi^2$

9. Assume $\vec{v} = \langle A, B, C \rangle$

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle,\ a \le t \le b$

$\int_C \vec{v}\cdot d\vec{r} = \int_a^b A\,x'(t) + B\,y'(t) + C\,z'(t)\,dt$

$= A\,x(t) + B\,y(t) + C\,z(t) \Big|_a^b$

$= \langle A, B, C\rangle \cdot \langle x(t), y(t), z(t)\rangle \Big|_a^b$

$= \vec{v}\cdot\vec{r}(t)\Big|_a^b$

$= \vec{v}\cdot\vec{r}(b) - \vec{v}\cdot\vec{r}(a)$

$= \vec{v}\cdot[\vec{r}(b) - \vec{r}(a)]$

Second-Half.

1. The vector field is defined on the region $\{(x,y)\in\mathbb{R}^2 \mid y>0\}$, which is simply connected

$$\frac{\partial}{\partial y}(\ln y + 2xy^3) = \frac{1}{y} + 6xy^2$$

$$\frac{\partial}{\partial x}(3x^2y^2 + \frac{x}{y}) = 6xy^2 + \frac{1}{y}.$$

So $\frac{\partial}{\partial y}(\ln y + 2xy^3) = \frac{\partial}{\partial x}(3x^2y^2 + \frac{x}{y})$

we conclude the vector field is conservative

2. Assume $\vec{F}(x,y) = \langle xy^2, x^2y\rangle = \nabla f$.

Then: $\begin{cases} \frac{\partial f}{\partial x} = xy^2 & \cdots ① \\ \frac{\partial f}{\partial y} = x^2y & \cdots ② \end{cases}$

by ① $f(x,y) = \frac{x^2y^2}{2} + g(y)$, then take $\frac{\partial}{\partial y}$ on both sides:

$$\frac{\partial f}{\partial y}(x,y) = x^2y + g'(y)$$

Compare with ②. we see $g'(y)=0$ so $g(y)=C$, a constant

So $f(x,y) = \frac{x^2y^2}{2} + C$

In particular, we can choose $f(x,y) = \frac{x^2y^2}{2}$

3. The vector field is defined on $\mathbb{R}^2$. $\mathbb{R}^2$ is simply connected.

$$\vec{F}(x,y) = \langle 2xe^{-y},\ 2y - x^2e^{-y} \rangle$$

$$\frac{\partial}{\partial y}(2xe^{-y}) = -2xe^{-y} = \frac{\partial}{\partial x}(2y - x^2e^{-y})$$

so $\vec{F}$ is conservative. hence $\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy$ is independent of the path.

We are going to find a potential function f of $\vec{F}$ to evaluate the line integral:

if $\nabla f = \langle 2xe^{-y},\ 2y - x^2e^{-y} \rangle$

then
$$\begin{cases} \dfrac{\partial f}{\partial x} = 2xe^{-y} & \text{--- ①} \\ \dfrac{\partial f}{\partial y} = 2y - x^2e^{-y} & \text{--- ②} \end{cases}$$

by ①. we get $f(x,y) = x^2e^{-y} + g(y)$. then take $\frac{\partial}{\partial y}$ on both sides

$$\frac{\partial f}{\partial y} = -x^2e^{-y} + g'(y).$$

Compare with ② we get $g'(y) = 2y$. so $g(y) = y^2 + C$

we may choose $g(y) = y^2$.

so $f(x,y) = x^2e^{-y} + y^2$

$$\int_C 2xe^{-y}dx + (2y - x^2e^{-y})dy = f(2,0) - f(1,0) = 4 - 1 = 3$$

4. We apply the Green's Theorem:

$$\oint_C (y+\sqrt{x})dx+(2x+\cos y^2)dy = \iint_D \frac{\partial(2x+\cos y^2)}{\partial x} - \frac{\partial(y+\sqrt{x})}{\partial y}\, dA$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 2-1 \, dy dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \, dy dx$$

$$= \frac{1}{3}$$

5. We apply the Green's Theorem:

$$\oint_C y^4 dx + 2xy^3 dy = \iint_D \frac{\partial(2xy^3)}{\partial x} - \frac{\partial y^4}{\partial y}\, dA$$

$$= \iint_D 2y^3 - 4y^3 \, dA$$

$$= -2\iint_D y^3 \, dA$$

Now we do a change of variable $x=\sqrt{2}u$, $y=v$.

Then the corresponding region on uv-plane is enclosed by

$(\sqrt{2}u)^2 + 2\cdot v^2 = 2$ i.e. $u^2+v^2=1$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{vmatrix} = \sqrt{2}$$

So $$-2\iint_D y^3 dA = -2\iint_{u^2+v^2\le 1} v^3 \cdot \sqrt{2}\, du dv = -2\int_0^{2\pi}\int_0^1 (r\sin\theta)^3 \cdot \sqrt{2}\, r dr d\theta$$

$$= -2\sqrt{2}\int_0^{2\pi}\int_0^1 r^4 \sin^3\theta \, dr d\theta$$

$$= -2\sqrt{2}\left(\int_0^{2\pi} \sin^3\theta \, d\theta\right)\left(\int_0^1 r^4 dr\right)$$

$$= 0$$

6.

$$\oint_C \vec{F}(x,y)\cdot d\vec{r} = \oint_C x\,dx + (x^3+3xy^2)\,dy$$

$$= \iint_D \frac{\partial(x^3+3xy^2)}{\partial x} - \frac{\partial x}{\partial y}\, dA$$

$$= \iint_D 3x^2+3y^2\, dA$$

$$= \int_0^{\pi}\int_0^2 3r^2\cdot r\,dr\,d\theta$$

$$= 12\pi$$

7.

$$\text{Curl}\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin yz & \sin zx & \sin xy \end{vmatrix} = \left\langle \frac{\partial \sin xy}{\partial y} - \frac{\partial \sin zx}{\partial z}, \frac{\partial \sin yz}{\partial z} - \frac{\partial \sin xy}{\partial x}, \frac{\partial \sin zx}{\partial x} - \frac{\partial \sin yz}{\partial y} \right\rangle$$

$$= \langle x\cos xy - x\cos zx,\ y\cos yz - y\cos xy,\ z\cos zx - z\cos yz \rangle$$

$$\text{div}\,\vec{F} = \frac{\partial \sin yz}{\partial x} + \frac{\partial \sin zx}{\partial y} + \frac{\partial \sin xy}{\partial z} = 0$$

8. Let $\vec{F} = \langle f_1, f_2, f_3 \rangle$. $\vec{G} = \langle g_1, g_2, g_3 \rangle$

$$\text{div}(\vec{F}\times\vec{G}) = \vec{\nabla}\cdot(\vec{F}\times\vec{G}) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$= \frac{\partial}{\partial x}(f_2 g_3 - f_3 g_2) + \frac{\partial}{\partial y}(f_3 g_1 - f_1 g_3) + \frac{\partial}{\partial z}(f_1 g_2 - f_2 g_1)$$

$$= f_2 \frac{\partial g_3}{\partial x} + g_3 \frac{\partial f_2}{\partial x} - f_3 \frac{\partial g_2}{\partial x} - g_2 \frac{\partial f_3}{\partial x}$$

$$+ f_3 \frac{\partial g_1}{\partial y} + g_1 \frac{\partial f_3}{\partial y} - f_1 \frac{\partial g_3}{\partial y} - g_3 \frac{\partial f_1}{\partial y}$$

$$+ f_1 \frac{\partial g_2}{\partial z} + g_2 \frac{\partial f_1}{\partial z} - f_2 \frac{\partial g_1}{\partial z} - g_1 \frac{\partial f_2}{\partial z}$$

On the other hand,

$$\vec{G}\cdot \text{curl}\vec{F} - \vec{F}\,\text{curl}\vec{G} = \begin{vmatrix} g_1 & g_2 & g_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} - \begin{vmatrix} f_1 & f_2 & f_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g_1 & g_2 & g_3 \end{vmatrix}$$

$$= g_1\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + g_2\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + g_3\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$- f_1\left(\frac{\partial g_3}{\partial y} - \frac{\partial g_2}{\partial z}\right) - f_2\left(\frac{\partial g_1}{\partial z} - \frac{\partial g_3}{\partial x}\right) - f_3\left(\frac{\partial g_2}{\partial x} - \frac{\partial g_1}{\partial y}\right)$$

$$= g_1\frac{\partial f_3}{\partial y} - g_1\frac{\partial f_2}{\partial z} + g_2\frac{\partial f_1}{\partial z} - g_2\frac{\partial f_3}{\partial x} + g_3\frac{\partial f_2}{\partial x} - g_3\frac{\partial f_1}{\partial y}$$

$$- f_1\frac{\partial g_3}{\partial y} + f_1\frac{\partial g_2}{\partial z} - f_2\frac{\partial g_1}{\partial z} + f_2\frac{\partial g_3}{\partial x} - f_3\frac{\partial g_2}{\partial x} + f_3\frac{\partial g_1}{\partial y}$$

Comparing the terms, we conclude

$$\text{div}(\vec{F}\times\vec{G}) = \vec{G}\cdot\text{Curl}\vec{F} - \vec{F}\,\text{Curl}\vec{G}$$

9. We know $\text{div}(\text{Curl}\,\vec{F}) = 0$

$$\text{div}(\langle x\sin y, \cos y, z-xy\rangle) = \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y) + \frac{\partial}{\partial z}(z-xy)$$

$$= \sin y - \sin y + 1$$

$$= 1 \neq 0$$

So this vector field cannot be $\text{Curl}\vec{F}$ for any $\vec{F}$.