Homework IV Solution

First-Half

1. (a) $$\int_1^4\int_0^2 6x^2y-2x\,dydx = \int_1^4 3x^2y^2-2xy\Big|_{y=0}^{y=2}dx$$
$$= \int_1^4 12x^2-4x\,dx$$
$$= 222$$

(b) $$\int_1^3\int_1^5 \frac{\ln y}{xy}\,dy\,dx = \int_1^3 \frac{1}{x}dx \int_1^5 \frac{\ln y}{y}dy$$
$$= \left(\ln x\Big|_{x=1}^{x=3}\right)\left(\frac{(\ln y)^2}{2}\Big|_{y=1}^{y=5}\right)$$
$$= \frac{(\ln 3)(\ln 5)^2}{2}$$

2. $$\iint_R x\sin(x+y)\,dA = \int_0^{\frac{\pi}{5}}\int_0^{\frac{\pi}{3}} x\sin(x+y)\,dydx$$
$$= \int_0^{\frac{\pi}{5}} -x\cos(x+y)\Big|_{y=0}^{y=\frac{\pi}{3}}dx$$
$$= \int_0^{\frac{\pi}{5}} -x\cos\left(x+\frac{\pi}{3}\right)+x\cos x\,dx$$
$$= \int_0^{\frac{\pi}{5}} -x\left(\cos x\cos\frac{\pi}{3}-\sin x\sin\frac{\pi}{3}\right)+x\cos x\,dx$$
$$= \int_0^{\frac{\pi}{5}} -\frac{1}{2}x\cos x+\frac{\sqrt{3}}{2}x\sin x+x\cos x\,dx$$
$$= \int_0^{\frac{\pi}{5}} x\left(\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x\right)dx$$
$$= \int_0^{\frac{\pi}{5}} x\sin\left(x+\frac{\pi}{6}\right)dx$$
$$= -\int_0^{\frac{\pi}{5}} x\,d\cos\left(x+\frac{\pi}{6}\right)$$
$$= -x\cos\left(x+\frac{\pi}{6}\right)\Big|_0^{\frac{\pi}{5}} + \int_0^{\frac{\pi}{5}}\cos\left(x+\frac{\pi}{6}\right)dx$$
$$= -\frac{\pi}{5}\cos\frac{11}{30}\pi - \sin\frac{11}{30}\pi + \frac{1}{2}$$

3. $\iint_R y e^{-xy} dA = \int_0^3 \int_0^2 y e^{-xy} dx\, dy$

$= \int_0^3 -e^{-xy} \Big|_{x=0}^{x=2} dy$

$= \int_0^3 -e^{-2y} + 1 \, dy$

$= \frac{1}{2} e^{-2y} + y \Big|_{y=0}^{y=3}$

$= \frac{1}{2}(e^{-6} - 1) + 3$

$= \frac{e^{-6}}{2} + \frac{5}{2}$

4. $4x + 6y - 2z + 15 = 0 \Rightarrow z = \frac{4x+6y+15}{2} = 2x + 3y + \frac{15}{2}$.

The volume is $\int_{-1}^{2} \int_{1}^{2} 2x + 3y + \frac{15}{2} \, dy\, dx$.

$= \int_{-1}^{2} 2xy + \frac{3}{2}y^2 + \frac{15}{2} y \Big|_{y=1}^{y=2} dx$

$= \int_{-1}^{2} 2x + 12 \, dx$

$= x^2 + 12x \Big|_{x=-1}^{x=2}$

$= 39$

5 (a). $\int_0^4 \int_0^{\sqrt{y}} xy^2 \, dx\, dy = \int_0^4 \frac{1}{2} x^2 y^2 \Big|_{x=0}^{x=\sqrt{y}} dy = \int_0^4 \frac{y^3}{2} dy = \frac{y^4}{8} \Big|_{y=0}^{y=4} = 32$

(b). $\int_0^1 \int_0^{e^v} \sqrt{1+e^v} \, dw\, dv = \int_0^1 \sqrt{1+e^v} \cdot e^v dv$

$= \int_0^1 \sqrt{1+e^v} \, d e^v$

$= \int_0^1 \sqrt{1+e^v} \, d(1+e^v)$

$= \frac{2}{3}(1+e^v)^{\frac{3}{2}} \Big|_{v=0}^{v=1}$

$= \frac{2}{3}(1+e)^{\frac{3}{2}} - \frac{2}{3} \cdot 2\sqrt{2}$

6. (a) $\iint_D y^2 \, dA = \int_{-1}^{1} \int_{-y-2}^{y} y^2 \, dx\,dy = \int_{-1}^{1} y^2 (2y+2)\,dy = \frac{4}{3}$

(b) $\iint_D x^3 \, dA = \int_1^e \int_0^{\ln x} x^3 \, dy\,dx = \int_1^e x^3 \ln x \, dx$

$$= \frac{1}{4}\int_1^e \ln x \, d x^4$$

$$= \frac{1}{4}\left(x^4 \ln x \Big|_1^e - \int_1^e x^4 \, d \ln x \right.$$

$$= \frac{1}{4}\left(e^4 - \int_1^e x^3 \, dx \right)$$

$$= \frac{1}{4}\left(e^4 - \frac{e^4}{4} + \frac{1}{4} \right)$$

$$= \frac{3e^4+1}{16}$$

7. (a) $\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} f(x,y)\,dx\,dy = \int_0^{\sqrt{\pi}} \int_0^{x} f(x,y)\,dy\,dx$

(b) $\int_0^4 \int_{\sqrt{x}}^2 f(x,y)\,dy\,dx = \int_0^2 \int_0^{y^2} f(x,y)\,dy\,dx$

8. $\iint_D x^2 y \, dA = \int_0^{\pi} \int_0^5 (r\cos\theta)^2 (r\sin\theta) \cdot r \, dr\, d\theta$

$$= \int_0^{\pi} \int_0^5 r^4 \cos^2\theta \sin\theta \, dr\, d\theta$$

$$= \left(\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \right)\left(\int_0^5 r^4 \, dr \right)$$

$$= \left(-\int_0^{\pi} \cos^2\theta \, d\cos\theta \right) \cdot \left(\frac{r^5}{5} \Big|_{r=0}^{r=5} \right)$$

$$= \left(-\frac{\cos^3\theta}{3} \Big|_{\theta=0}^{\theta=\pi} \right) \cdot 625$$

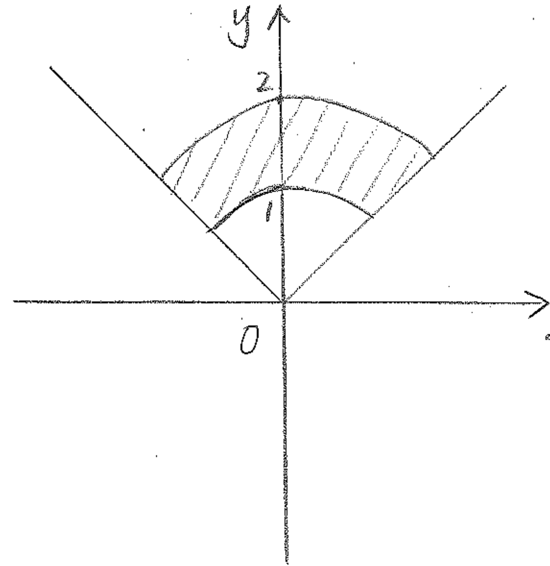
$$= \frac{2}{3} \cdot 625$$

$$= \frac{1250}{3}$$

9. $\int_{\frac{\pi}{4}}^{\frac{3}{4}\pi}\int_1^2 r\,dr\,d\theta = \int_{\frac{\pi}{4}}^{\frac{3}{4}\pi} \frac{r^2}{2}\Big|_{r=1}^{r=2} d\theta$

$= \frac{3}{2}\cdot\frac{\pi}{2}$

$= \frac{3}{4}\pi$



10. $\int_{-3}^{3}\int_0^{\sqrt{9-x^2}} \sin(x^2+y^2)\,dy\,dx = \int_0^{\pi}\int_0^3 (\sin r^2)\, r\,dr\,d\theta$

$= \frac{1}{2}\int_0^{\pi} 1\,d\theta \cdot \int_0^3 \sin r^2\, dr^2$

$= \frac{\pi}{2}\cdot\left(-\cos r^2\Big|_{r=0}^{r=3}\right)$

$= \frac{\pi}{2}\left(1-\cos 9\right)$.

Second-Half

1. $$\int_0^1\int_y^1\int_0^{xy} y\,dz\,dx\,dy = \int_0^1\int_y^1 xy^2\,dx\,dy = \int_0^1 \frac{x^2y^2}{2}\Big|_{x=y}^{x=1} dy$$
$$= \int_0^1 \frac{y^2}{2} - \frac{y^4}{2}\,dy$$
$$= \frac{1}{2}\left(\frac{y^3}{3} - \frac{y^5}{5}\right)\Big|_{y=0}^{y=1}$$
$$= \frac{1}{15}$$

2. $$\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{x+y} xy\,dz\,dy\,dx = \int_0^1\int_{x^2}^{\sqrt{x}} xy(x+y)\,dy\,dx$$
$$= \int_0^1\int_{x^2}^{\sqrt{x}} x^2y + xy^2\,dy\,dx$$
$$= \int_0^1 \frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\Big|_{y=x^2}^{y=\sqrt{x}}\,dx$$
$$= \int_0^1 \frac{1}{2}x^3 + \frac{1}{3}x^{\frac{5}{2}} - \frac{1}{2}x^6 - \frac{1}{3}x^7\,dx$$
$$= \frac{1}{8}x^4 + \frac{2}{21}x^{\frac{7}{2}} - \frac{1}{14}x^7 - \frac{1}{24}x^8\Big|_0^1$$
$$= \frac{3}{28}$$

3. $$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+z^2}^{8-x^2-z^2} 1\,dy\,dz\,dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 8-2x^2-2z^2\,dz\,dx$$
$$= \int_0^{2\pi}\int_0^2 (8-2r^2)\,r\,dr\,d\theta$$
$$= \int_0^{2\pi} 1\,d\theta \int_0^2 8r-2r^3\,dr$$
$$= 16\pi$$

4.

$$\iiint_E f(x,y,z)\,dV = \int_{-2}^{2}\int_{x^2}^{4}\int_0^{\frac{4-y}{2}} f(x,y,z)\,dz\,dy\,dx$$

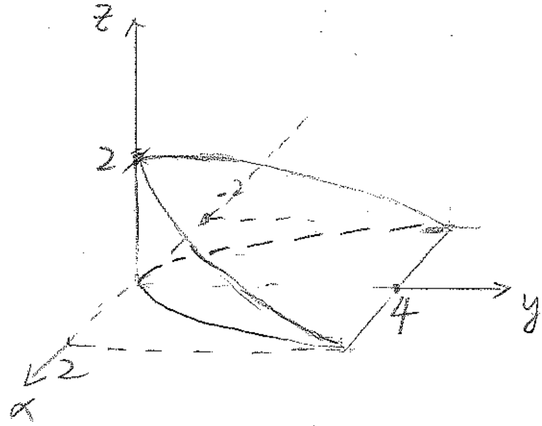
$$= \int_0^4\int_{-\sqrt{y}}^{\sqrt{y}}\int_0^{\frac{4-y}{2}} f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^4\int_0^{\frac{4-y}{2}}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^2\int_0^{4-2z}\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_0^{\frac{4-x^2}{2}}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^2\int_{-\sqrt{4-2z}}^{\sqrt{4-2z}}\int_{x^2}^{4-2z} f(x,y,z)\,dy\,dx\,dz$$



5.

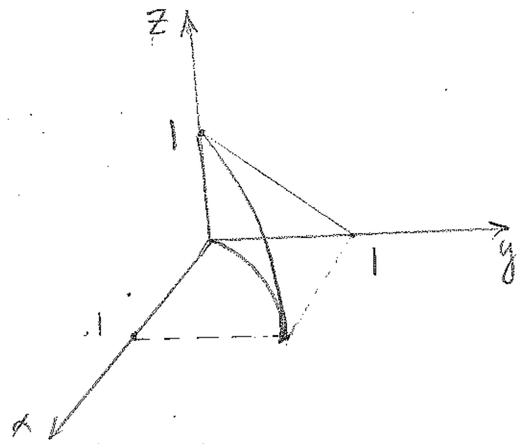
$$\int_0^1\int_{\sqrt{x}}^{1}\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1\int_0^{(z-1)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz$$



6. $\begin{cases} x = 2u + v \\ y = u + 2v \end{cases} \Longleftrightarrow \begin{cases} u = \dfrac{2x - y}{3} \\ v = \dfrac{2y - x}{3} \end{cases}$

the triangle in xy-plane is bounded by the straight lines

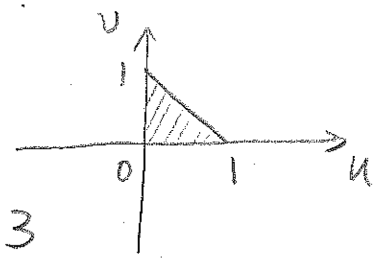
$y = 2x$, $y = \frac{1}{2}x$ and $y = -x + 3$

their images in uv-planes are

$u + 2v = 2(2u + v)$, $u + 2v = \frac{1}{2}(2u + v)$ and $u + 2v = -(2u + v) + 3$

i.e $u = 0$, $v = 0$ and $u + v = 1$.

so the corresponding region on the uv-plane is the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$



The Jacobian is $\dfrac{\partial x \partial y}{\partial u \partial y} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$

So $\displaystyle\iint_R (x - 3y)\, dA = \int_0^1 \int_0^{1-u} \big((2u + v) - 3(u + 2v)\big) \cdot 3 \, dv\, du$

$\displaystyle = \int_0^1 \int_0^{1-u} -3(u + 5v)\, dv\, du$

$= -3$

7. The region in the uvw-space is bounded by

$$\left(\frac{au}{a}\right)^2+\left(\frac{bv}{b}\right)^2+\left(\frac{cw}{c}\right)^2=1.$$

i.e. $u^2+v^2+w^2=1$.

The Jacobian is

$$\frac{\partial x\,\partial y\,\partial z}{\partial u\,\partial v\,\partial w}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w}\\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}a & 0 & 0\\ 0 & b & 0\\ 0 & 0 & c\end{vmatrix}=abc$$

$$\text{so } \iiint_E 1\,dV=\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} 1\cdot abc\,dw\,dv\,du$$

$$=abc\cdot\int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}}\int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} dw\,dv\,du$$

$$=abc\cdot\frac{4}{3}\pi$$

8. Let $\begin{cases}u=x+y\\ v=x-y\end{cases} \Longleftrightarrow \begin{cases}x=\frac{u+v}{2}\\ y=\frac{u-v}{2}\end{cases}$

the region R on xy-plane is bounded by the straight lines

$x-y=0,\ x-y=2,\ x+y=0,\ x+y=3$

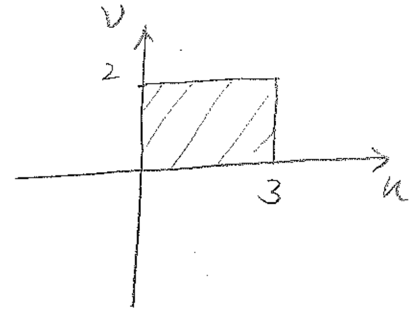
so the corresponding region on uv-plane is bounded by

$$\frac{u+v}{2}-\frac{u-v}{2}=0,\quad \frac{u+v}{2}-\frac{u-v}{2}=2,\quad \frac{u+v}{2}+\frac{u-v}{2}=0,\quad \frac{u+v}{2}+\frac{u-v}{2}=3.$$

i.e. $v=0,\ v=2,\ u=0,\ u=3$

Jacobian is

$$\frac{\partial x \partial y}{\partial u \partial v} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$



So

$$\iint_R (x+y) e^{x^2-y^2} dA = \int_0^3 \int_0^2 u\, e^{uv} \cdot \frac{1}{2}\, dv\, du$$

$$= \frac{1}{2} \int_0^3 e^{uv} \Big|_{v=0}^{v=2} du$$

$$= \frac{1}{2} \int_0^3 e^{2u} - 1\, du$$

$$= \frac{1}{2} \left(\frac{1}{2} e^{2u} - u \right) \Big|_0^3$$

$$= \frac{e^6 - 7}{4}.$$