# Homework III Solution

1. Let
$$\begin{cases} \frac{\partial f}{\partial x} = 6xy - 12x = 0 \\ \frac{\partial f}{\partial y} = 3y^2 + 3x^2 - 12y = 0 \end{cases}$$
   We get the critical points are: $(0,0),(0,4),(2,2),(-2,2)$.

   $\frac{\partial^2 f}{\partial x^2}(x,y) = 6y - 12$, $\frac{\partial^2 f}{\partial y^2}(x,y) = 6y - 12$, $\frac{\partial^2 f}{\partial x \partial y}(x,y) = 6x$

   So $D(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y)\frac{\partial^2 f}{\partial y^2}(x,y) - (\frac{\partial^2 f}{\partial x \partial y}(x,y))^2 = (6y-12)^2 - 6x^2$

   At $(0,0)$: $D(0,0) = 144 > 0$, $\frac{\partial^2 f}{\partial x^2}(0,0) = -12 < 0$, so it's a local maximum

   At $(0,4)$: $D(0,4) = 144 > 0$, $\frac{\partial^2 f}{\partial x^2}(0,0) = 12 > 0$, so it's a local minimum

   At $(2,2)$: $D(2,2) = -24 < 0$, so it's a saddle point

   At $(-2,2)$: $D(-2,2) = -24 < 0$, so it's a saddle point

2. We first look for critical points in the interior:
$$\begin{cases} \frac{\partial f}{\partial x} = 2x - 2 = 0 \\ \frac{\partial f}{\partial y} = 2y = 0 \end{cases}$$
   We get the critical point is $(1,0)$. $f(1,0) = -1$.

   Next we consider the boundary points. On the segment $(0,2)--(0,-2)$, $f(x,y) = f(0,y) = y^2$, $-2 \le y \le 2$. So the maximum is at $(0,2)$ and $(0,-2)$, with value $f(0,2) = f(0,-2) = 4$. The minimum is at $(0,0)$, with value $f(0,0) = 0$.

   On the segment $(0,2)--(2,0)$, $f(x,y) = f(x,2-x) = 2(x-1)(x-2)$, $0 \le x \le 2$ so the maximum is at $(0,2)$, with value $f(0,2) = 4$. The minimum is at $(\frac{3}{2},\frac{1}{2})$, with value $f(\frac{3}{2},\frac{1}{2}) = -\frac{1}{2}$.

On the segment $(0,-2)--(2,0)$, $f(x,y) = f(x,x-2) = 2(x-1)(x-2)$, $0 \leq x \leq 2$, so the maximum is at $(0,-2)$, with value $f(0,-2) = 4$. The minimum is at $(\frac{3}{2},-\frac{1}{2})$, with value $f(\frac{3}{2},-\frac{1}{2}) = -\frac{1}{2}$.

So we conclude on this region, the maximum value is 4, obtained at $(0,2)$ and $(0,-2)$. The minimum value is $-1$, obtained at $(1,0)$.

3. The profit function is given by

$$\pi(x,y) = 12x+6y-(x^2-2xy+2y^2-20x-10y+514) = -x^2-2y^2+2xy+32x+16y-514$$

Let

$$\begin{cases} \frac{\partial f}{\partial x} = -2x+2y+32 = 0 \\ \frac{\partial f}{\partial y} = -4y+2x+16 = 0 \end{cases}$$

We get the critical point is $(40,24)$.

$\frac{\partial^2 f}{\partial x^2} = -2$, $\frac{\partial^2 f}{\partial x \partial y} = 2$, $\frac{\partial^2 f}{\partial y^2} = -4$, so $D(40,24) = (-2)\times(-4)-2^2 = 4 > 0$, and $\frac{\partial^2 f}{\partial x^2}(40,24) = -2 < 0$, so it's a local maximum.

We conclude the profit is maximized at $(40,24)$.

4. Assume the dimensions are $x,y,z$. Its volume is $V = xyz$, and we will maximize it under the constraint that the sum of its 12 edges is $c$, i.e. $4x+4y+4z = c$.

Let $g(x,y,z) = 4x+4y+4z$

So we solve for the following equations

$$\begin{cases} \nabla V = \lambda \nabla g \\ g(x,y,z) = c \end{cases}$$

which become

$$\begin{cases} yz = 4\lambda \\ xz = 4\lambda \\ xy = 4\lambda \\ 4x+4y+4z = c \end{cases}$$

The only positive solution is $(\frac{c}{12},\frac{c}{12},\frac{c}{12})$. So the maximum volume is achieved when it is a cube with all edges have length $\frac{c}{12}$

5. We will maximize $P = 2pq + 2pr + 2rq$, with constraint $g(p.q.r) = p + q + r = 1$.

$$\begin{cases} \nabla P = \lambda \nabla g \\ p + q + r = 1 \end{cases}$$

i.e.

$$\begin{cases} 2(q + r) = \lambda \\ 2(p + r) = \lambda \\ 2(p + q) = \lambda \\ p + q + r = 1 \end{cases}$$

The solution is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, with $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{2}{3}$. The way to test whether $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is a maximum or minimum is to compare with any other point on the constraint, say $(1, 0, 0)$: $P(1, 0, 0) = 0 < f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, so $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ cannot be minimum, it has to be maximum. We conclude $P \leq \frac{2}{3}$.

6. We will maximize the square of the area of the triangle, which is

$$f(a, b, c) = s(s - a)(s - b)(s - c)$$

under the constraint $g(a, b, c) = a + b + c = 2s$

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(a, b, c) = 2s \end{cases}$$

which are indeed

$$\begin{cases} -s(s - b)(s - c) = \lambda \\ -s(s - a)(s - c) = \lambda \\ -s(s - a)(s - b) = \lambda \\ a + b + c = 2s \end{cases}$$

The solution is $a = b = c = \frac{2s}{3}$, so the largest possible area is

$$\sqrt{f(\frac{2s}{3}, \frac{2s}{3}, \frac{2s}{3})} = \frac{s^2}{3\sqrt{3}} = \frac{s^2}{9}\sqrt{3}$$

7. We will find the maximum and minimum of $V = xyz$ with the constraints $2(xy+xz+yz) = 1500$ and $4(x+y+z) = 200$, i.e. $g(x,y,z) = xy + xz + yz = 750$ and $h(x,y,z) = x + y + z = 50$.

   So we solve the following system of equations:

$$\begin{cases} \nabla V = \lambda \nabla g + \mu \nabla h \\ xy + xz + yz = 750 \\ x + y + z = 50 \end{cases}$$

   i.e.

$$\begin{cases} yz = \lambda(y+z) + \mu \\ xz = \lambda(x+z) + \mu \\ xy = \lambda(x+y) + \mu \\ xy + xz + yz = 750 \\ x + y + z = 50 \end{cases}$$

   From the first three equations we get:

$$\begin{cases} (y-x)z = \lambda(y-x) \\ (z-y)x = \lambda(z-y) \\ (x-z)y = \lambda(x-z) \end{cases}$$

   Suppose $x, y, z$ are three different numbers, we get $x = y = z = \lambda$, contradiction. So if $(x,y,z)$ is a solution, at least two of $x, y, z$ are equal. Since $x, y, z$ are symmetric, we may assume $x$ and $y$ are the edges of equal length, i.e. $x = y$. Then putting it into

$$\begin{cases} xy + xz + yz = 750 \\ x + y + z = 50 \end{cases}$$

   We get the solutions are $(\frac{50+5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}, \frac{50-10\sqrt{10}}{3})$ and $(\frac{50+5\sqrt{10}}{3}, \frac{50+5\sqrt{10}}{3}, \frac{50-10\sqrt{10}}{3})$. The volume for the former case is $\frac{2500}{27}(35-\sqrt{10})$, and the volume of the later is $\frac{2500}{27}(35+\sqrt{10})$. So the maximum of the volume is $\frac{2500}{27}(35+\sqrt{10})$ and the minimum of the volume is $\frac{2500}{27}(35-\sqrt{10})$.

8. We will maximize and minimize the function $f(x,y,z) = x^2 + y^2 + z^2$ under the constraints $g(x,y,z) = x + y + 2z = 2$, $h(x,y,z) = x^2 + y^2 - z = 0$

So we solve the following system of equations:

$$\begin{cases} \nabla f = \lambda \nabla g + \mu \nabla h \\ x + y + 2z = 2 \\ x^2 + y^2 - z = 0 \end{cases}$$

i.e.

$$\begin{cases} 2x = \lambda + 2\mu x \\ 2y = \lambda + 2\mu y \\ 2z = 2\lambda - \mu \\ x + y + 2z = 2 \\ x^2 + y^2 - z = 0 \end{cases}$$

The solutions are $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ when $\lambda = \frac{2}{3}$ and $\mu = \frac{1}{3}$, and $(-1, -1, 2)$ with $\lambda = \frac{10}{3}$ and $\mu = \frac{8}{3}$.

$f(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) = \frac{3}{4}$, and $f(-1, -1, 2) = 6$, so the maximum distance is $\sqrt{6}$, and the minimum distance is $\frac{\sqrt{3}}{2}$.

9. The area of the rectangle is

$$S = \frac{1}{2} ab \sin C = \frac{1}{2}(2r \sin A)(2r \sin B) \sin C = 2r^2 \sin A \sin B \sin C$$

And the constraint is $A+B+C = \pi$. So we let $g(A, B, C) = A+B+C$.

Consider the following equation:

$$\begin{cases} \nabla V = \lambda \nabla g \\ g(A, B, C) = \pi \end{cases}$$

which becomes

$$\begin{cases} 2r^2 \cos A \sin B \sin C = \lambda \\ 2r^2 \sin A \cos B \sin C = \lambda \\ 2r^2 \sin A \sin B \cos C = \lambda \\ A + B + C = \pi \end{cases}$$

The first two equations imply $\cos A \sin B = \sin A \cos B$, which further implies $\sin(A-B) = 0$, so $A = B$. Similarly, we can also prove $B = C$, so $A = B = C = \frac{\pi}{3}$.

So the largest possible area is $S = 2r^2 (\sin \frac{\pi}{3})^3 = \frac{3\sqrt{3}r^2}{4}$